Stability of radiation-pressure particle traps: an optical Earnshaw theorem

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We prove an optical radiation Earnshaw theorem: A small dielectric particle cannot be trapped by using only the scattering force of optical radiation pressure. A corollary is that the gradient or dipole force is necessary to any successful optical trap. We discuss the implications of the theorem for recent proposals for the optical trapping of neutral atoms.

We derive an optical Earnshaw theorem that states that it is impossible to trap a small dielectric particle at a point of stable equilibrium in free space by using only the scattering force of radiation pressure. The theorem is analogous to Earnshaw's theorem in electrostatics,[1] which states that it is impossible to trap a charged particle by using only electrostatic forces. We show explicitly that two recently proposed traps for neutral atoms using the scattering force are in fact unstable, in agreement with our general theorem. The implications of the theorem for the stable trapping and cooling of atoms are discussed.

It is well known that optical trapping of dielectric particles is possible by using the forces of radiation pressure from lasers.[2,3] These forces arise from the scattering of light momentum by the particles.[4] Micrometer-sized particles (in the Mie-scattering regime, $d \gtrsim \lambda$) have been optically levitated[5] by a single laser beam and trapped stably by two beams.[2,6]

Here the principal contribution to the light scattering comes from refraction of the incident light rays passing through the particle.[2] For a spherical Mie particle in a plane-wave beam, the scattering is symmetric and the force is in the direction of the incident Poynting vector. If the Mie particle is in a beam with a transverse gradient, the scattering is no longer symmetric and the force has an additional component transverse to the Poynting vector. With submicrometer-sized particles (in the Rayleigh-scattering regime, $d \ll \lambda$; this includes atoms) it is again convenient and useful to divide the total force into two components. One is called the scattering force.[2,7] It is proportional to the scattering cross section of the particle and for paraxial beams and scalar polarizability is in the direction of the Poynting vector. It is a nonconservative force resulting from the removal of momentum from the incident beam. The other is called the gradient force.[8–11] It arises from the interference of the scattered field with the incident field and is proportional to the in-phase component of the particle's polarizability. It attracts particles with positive polarizability to regions of high electric-field strength. The gradient force is a conservative force whose potential is the free energy[12] of the particle. Thus it can also be regarded as the electrostrictive force on the optically induced dipole of the particle in a field-intensity gradient.[13] For this reason the gradient force on Rayleigh particles is also referred to as the dipole force. Stable trapping of submicrometer dielectric particles has been observed in the standing-wave field of two beams.[14] Trapping of neutral atoms is more difficult and has not yet been accomplished, although the basic forces have been experimentally demonstrated.[15] In this case resonance can be used to increase the magnitude of the forces, but saturation effects limit the trap depth, and the heating effects of quantum fluctuations become important.

It is our thesis that the gradient force is essential to any trap for small particles. The strong velocity dependence of the scattering force has led to the useful concept of optical cooling[16] of atoms by light tuned below resonance, but the scattering force by itself cannot form a trap.

Before the recent proposals of Minogin[17] and Minogin and Javanainen[18] for atom traps based on the scattering force, all the experimentally demonstrated or proposed optical traps for neutral particles involved use of the gradient force. The earliest proposal for trapping atoms about a point of stable equilibrium[10] involved confining atoms to the intensity maxima of standing-wave fields. The light was tuned below resonance by $\gamma_N/2$ (half of the natural line width) so that the same beams would give trapping and optical cooling. This proposed trap is stable, in that the gradient force is restoring for arbitrary displacements from a point of maximum electric energy density. However, saturation of the atomic resonance limits the depth of this trap to an energy of about $h\gamma_N$, which is the same as the minimum kinetic energy to which the atoms can be cooled. Thus this trap is leaky, and in fact the same result seems to occur for any trap in which the same beam is used for both trapping and cooling.

To overcome these difficulties another class of trap was proposed[11] in which the trapping field was tuned much farther below resonance. This inhibits saturation and with the help of beam focusing allows the trap depth to be increased, in proportion to the detuning, by

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as much as 10^4 times. Since the detuning similarly decreases the optical cooling, the use of auxiliary damping beams[19,20] tuned closer to resonance was proposed for additional cooling. The dynamic Stark shift of the atomic resonance caused by the trapping field substantially reduces the effectiveness of the auxiliary damping beams, however, so this proposal has its own difficulties.

The purpose of the most recent trap proposals of Minogin[17] and of Minogin and Javanainen[18] was to circumvent all these difficulties by relying solely on the scattering force for stability. These workers suggested the use of four or six beams to provide simultaneous trapping and cooling. However, as we now show specifically for these proposed traps and in general for any trap, directions exist at which the scattering force points away from the trap, making it unstable. As Earnshaw's theorem in electrostatics is a direct result of div(**E**) = 0 in vacuum, so our theorem is a direct result of div(scattering force) = 0, a relation that applies so long as saturation effects are neglected.

These recent trap proposals make use of the scattering force in the far field of Gaussian laser beams, so let us first describe this force. Figure 1 shows the geometry. In the far field ($z \gg \pi w_0^2/\lambda$), the scattering force is proportional to the optical power hitting the particle and is directed normal to the spherical phase fronts, whose centers of curvature are at the beam waist. In the paraxial approximation, with $z \gg x$ or $z \gg y$, we can express the nonzero components of the scattering force **F** in cylindrical coordinates as

$$F_z = Kw^{-2} \exp(-2r^2/w^2), \tag{1}$$

$$F_r = F_z(r/z), \tag{2}$$

where the spot size $w = \lambda z/\pi w_0$ and K is a constant proportional to the particle's scattering cross section and to the total power in the light beam. Only the exponential factor distinguishes this force from the electrostatic force on a test charge that is due to a point charge located on the axis at the beam waist at $z = 0$. Otherwise the force is similarly directed and has the same $1/R^2$ dependence on position.

Consider now the four-beam trap of Minogen and Javanainan,[18] as illustrated in Fig. 2. Here four beams are tetrahedrally arranged in an attempt to form a trap

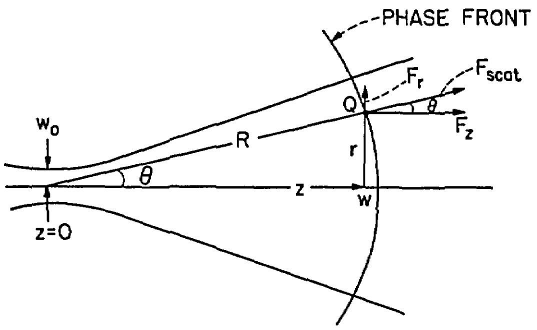

Fig. 1. Geometry of the scattering force on a particle placed at Q in the far field of a Gaussian beam with waist w_0 at $z = 0$.

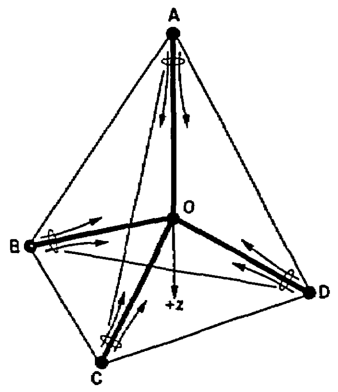

Fig. 2. Geometry of a proposed tetrahedral far-field scattering force atom trap, which we show to be unstable. The four beams waists are at A, B, C, and D.

at their intersection O. Now Earnshaw's theorem says that the analogous electrostatic trap with a tetrahedral arrangement of four charges located at A, B, C, and D is unstable. In the optical trap, as one proceeds along the z axis, for example, away from the intersection point O, the z-directed beam maintains its electrostatic analogy but the other beams, making an angle with the z axis, lose effectiveness because of the exponential falloff. Hence, as a particle proceeds out of the trap along the $+z$ axis, the restoring force that is due to the other beams is less than in the electrostatic case, which is already unstable. Indeed, if one adds the forces that are due to the four beams, one finds along the z axis in the neighborhood of the intersection ($x = y = z = 0$) the result that $F_z = K(4z/3w^2)^2 + \ldots$. Not only is there no linear restoring force, but the force always points in the direction of positive z.

Similar considerations apply to Minogin's six-beam trap.[17] Here the beam waists are located along the x, y, and z coordinate axes at a distance l from the origin, and all six beams shine inward toward the trap situated at the origin. Again one finds that there is no linear restoring force. There is a cubic restoring force along the axes given by $F_i = -8Kx_i^3/lw^4$ (for the x_i axis), but the particle can escape along any of the [1, 1, 1] directions that make equal angles with the coordinate axes. If s is the distance away from the origin in one of these directions, one finds that $F_s = 16Ks^3/3lw^4$. Again the trap is unstable.

These specific results can be generalized to any similar trap that uses the far fields of Gaussian beams by noting from Eqs. (1) and (2) that div(**F**) = 0. Such a divergenceless force can be represented by continuous lines, which must leave any volume that they enter, thus necessarily providing escape routes for the particles. Any sum of such forces has the same property.

We can proceed further to prove in general that the scattering force has zero divergence for small dipolar particles of arbitrary shape and properties and for optical fields of arbitrary geometry, provided only that the particle's dipole is linearly related to the field. We do not consider saturation effects, but they are detrimental to the formation and stability of any trap. The nonrelativistic Lorentz force exerted on a small neutral particle by the light field in the electric-dipole approximation is[20] $\mathbf{F}_l = (1/2)\mathrm{Re}\{\hat{x}_j \boldsymbol{\mu}^* \cdot \partial\mathbf{E}/\partial x_j\}$, where $\boldsymbol{\mu}$ is the particle's dipole moment, $\mathbf{E}$ is the electric field of the incident wave, $\hat{x}_j$ are the Cartesian-coordinate unit vectors, and the sum over j is implied. A single frequency $[\mathbf{E} \propto \exp(-i\omega t)]$ is assumed. This force is the sum of the electric force on the dipole moment of the particle and the magnetic force on the associated current. For quantum systems such as atoms, the same expression applies[20] for the expectation value of the force if the light field is coherent and $\boldsymbol{\mu}$ is the expectation value for the atomic dipole. The trapping-force field is this force evaluated for a stationary atom.

If $\boldsymbol{\mu}$ and $\mathbf{E}$ are related linearly, then $\boldsymbol{\mu} = \chi \cdot \mathbf{E}$, where χ is an arbitrary constant polarizability tensor. Now any such tensor can be developed as $\chi = \chi' + i\chi''$, where χ' and χ'' are Hermitian. In the absence of a dc magnetic field χ' and χ'' are symmetric and therefore real, but that is not necessary to our argument. On inserting this development for $\boldsymbol{\mu}$ into the expression for $\mathbf{F}_l$, we obtain $\mathbf{F}_l = \mathbf{F}_g + \mathbf{F}_s$, where

$$\mathbf{F}_g = (1/4)\mathrm{grad}(\mathbf{E}^* \cdot \chi' \cdot \mathbf{E}), \tag{3}$$

$$\mathbf{F}_s = (1/2)\mathrm{Im}\{\hat{x}_j \mathbf{E}^* \cdot \chi'' \cdot \partial\mathbf{E}/\partial x_j\}. \tag{4}$$

These are the general expressions for the gradient force and the scattering force, respectively. One can see that the potential for the gradient force is just the ac analog of the electrostatic free energy $[-(1/2)\boldsymbol{\mu} \cdot \mathbf{E}]$ of a dipole.[12] The scattering force is intimately related to the rate of work done on the dipole by the incident field $[(\omega/2)\mathbf{E}^* \cdot \chi'' \cdot \mathbf{E}]$ and therefore to the absorption plus scattering cross section of the particle. In the case of scalar polarizability and paraxial radiation the scattering force is proportional to Poynting's vector. Using the wave equation $\nabla^2\mathbf{E} = -(\omega/c)^2\mathbf{E}$, one can quickly show from Eq. (4) that $\mathrm{div}(\mathbf{F}_s) = 0$. The optical Earnshaw theorem is thus proved in considerable generality. In contrast to the electrostatic case, we do not have $\mathrm{curl}(\mathbf{F}_s) = 0$, so some lines of the scattering force may circulate within some volume. Such vortex behavior has been shown in the similar case of the Poynting vector.[21] A trap, however, must have some volume where the force is inward over its whole surface, and this is impossible for the scattering force since $\mathrm{div}(\mathbf{F}_s) = 0$.

The major implication of the optical Earnshaw theorem and its corollary that gradient forces are necessary for traps is that to produce deep traps one must maximize the gradient contributions to the force. Traps that rely on maximizing the scattering forces to the neglect of gradient forces are necessarily flawed, as we have proven. The actual scattering-force traps of Minogen and Javanainen were tuned $\gamma_N/2$ below resonance for cooling purposes. This introduces a small inward gradient force, which for their geometry cannot compensate for the gross instability that is due to the scattering force in the unstable directions.

Finally, the traps based on maximizing the gradient force[11,19,20] present the best prospects for experimentally achieving all optical neutral-atom traps. For optical powers of ~1 W, trap depths $\sim 10^4\ h\gamma_N$ are achievable for sodium atoms, which confine atoms of velocities $\lesssim 2 \times 10^3$ cm/sec. Optical traps are therefore relatively shallow compared with ion traps.[22] However, recent experiments on optical cooling of sodium atomic beams using the scattering force[23] have produced slow atoms with velocity $\sim 4 \times 10^3$ cm/sec. Such slowing techniques may thus ultimately provide a means of filling atom traps. Cooling difficulties that are due to a dynamic Stark shift remain for highly detuned gradient traps. The suggested use of an additional Stark-shift-canceling beam[19,20] is still a possible solution to the cooling problem. However, as has been pointed out,[20] quantum heating can be reduced enough to give a 1-sec retention time for cold atoms in such traps in the absence of any cooling.

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