Bose-Einstein condensation

- Ideal Bose gas
- Weakly interacting homogenous Bose gas
- Inhomogeneous Bose gas
- Superfluid hydrodynamics

Ideal BEC

See thermodynamics textbooks

To remember:

(1) Whether BEC occurs or not depends on density of states:

Power law, depends on dimension and confinement

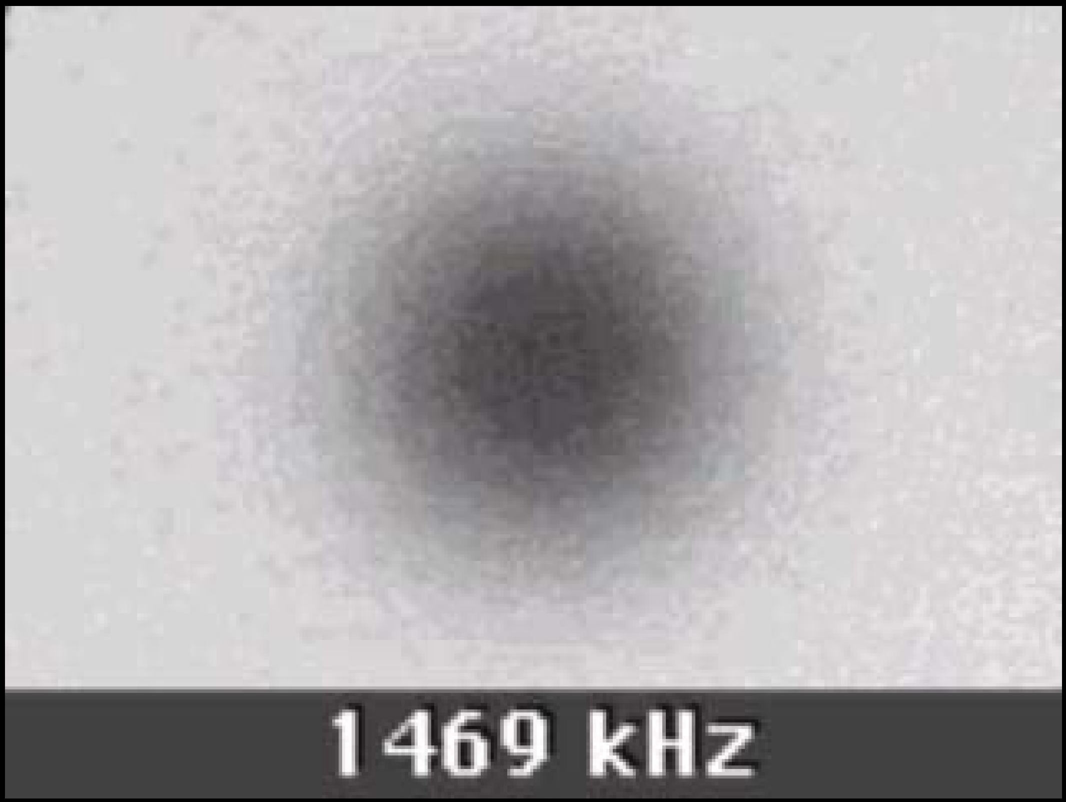
(2) Even for interacting BEC, normal component is described as ideal gas

T_C

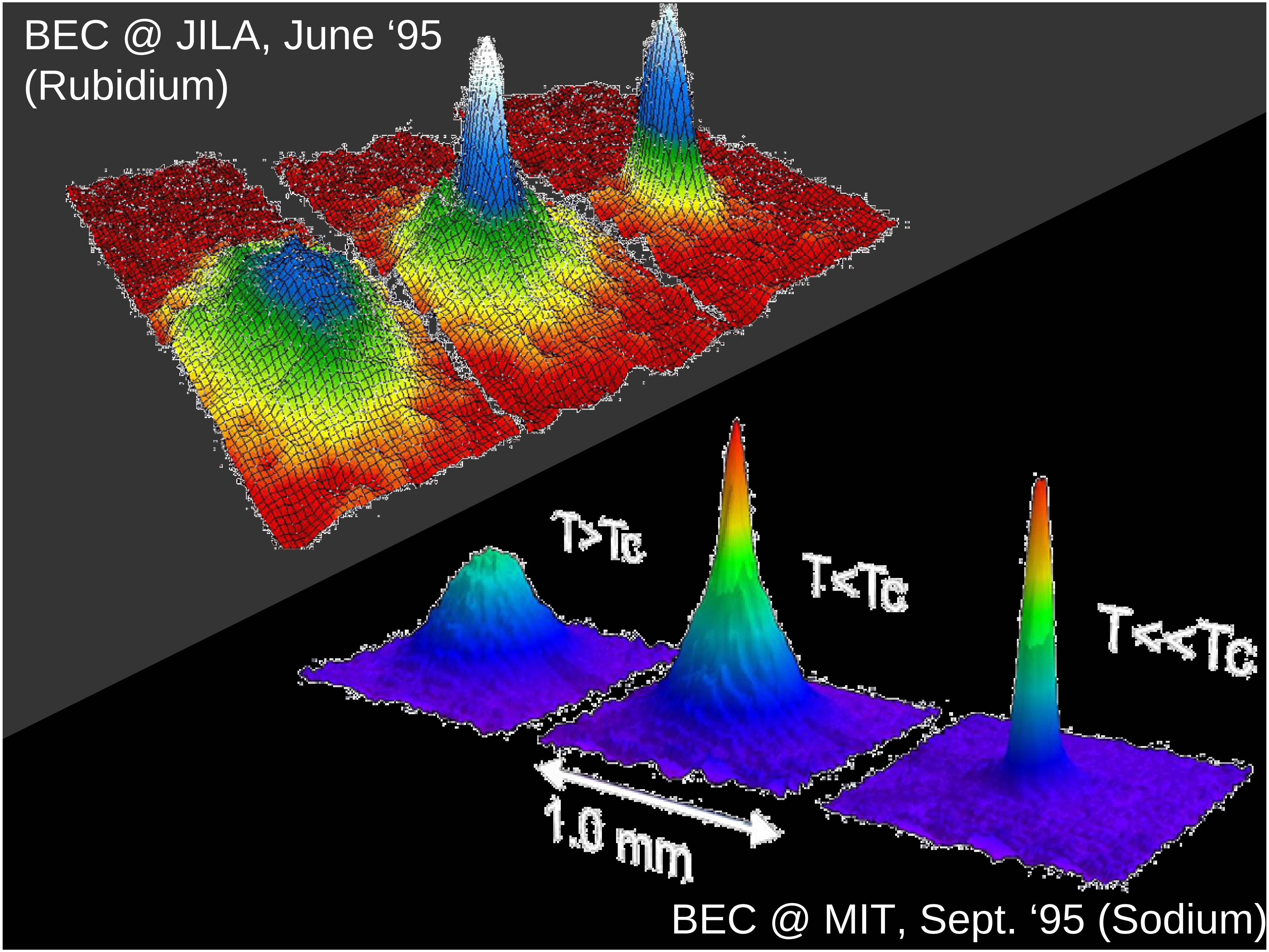
Condensate fraction

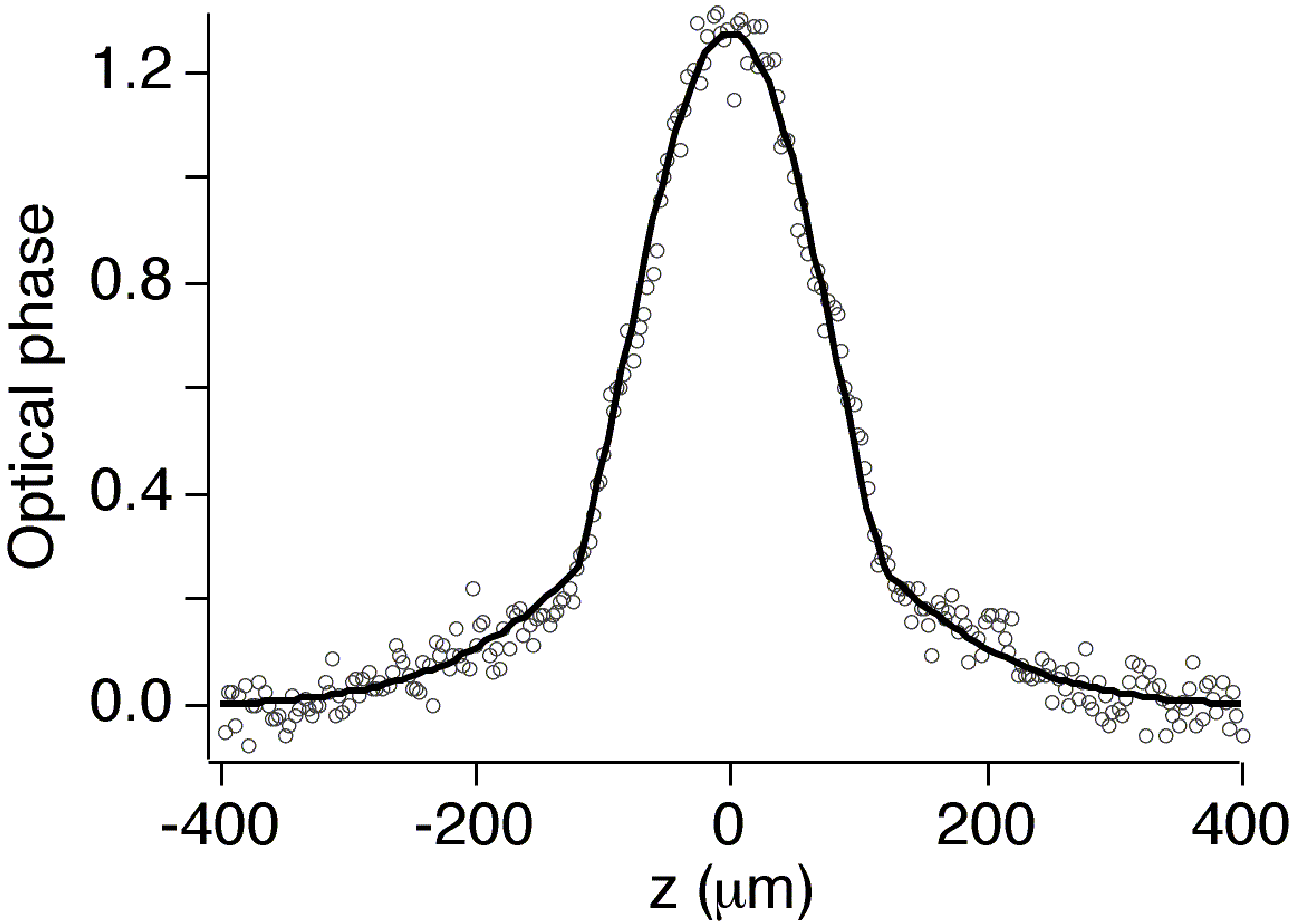
The shadow of a cloud of bosons
as the temperature is decreased

(Ballistic expansion for a fixed time-of-flight)

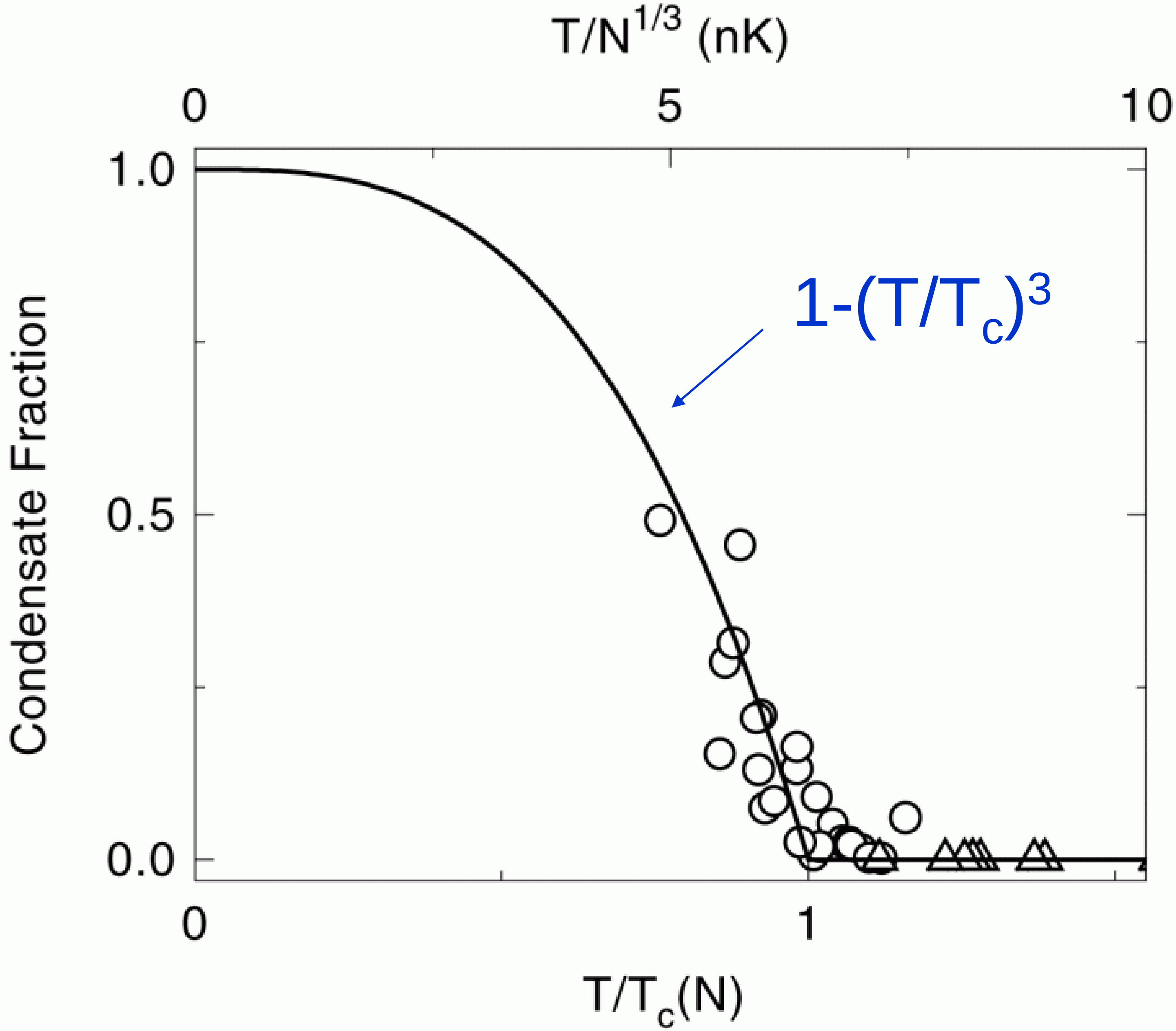


Temperature is linearly related to the rf frequency
which controls the evaporation


BEC @ JILA, June ‘95
(Rubidium)
T>Tc
T<Tc
T<<Tc
1.0 mm
BEC @ MIT, Sept. ‘95 (Sodium)



Optical phase
1.2
0.8
0.4
0.0
-400
-200
0
200
400
z (μm)


T/N^1/3 (nK)
0
5
10
1.0
0.5
0.0
Condensate Fraction
1-(T/Tc)^3
0
1
T/Tc(N)

Homogeneous BEC

Weakly interacting Bose gas at $T=0$

$$\hat{H}_1 = \frac{1}{2V} \sum U_q \, a^{+}_{p+q} \, a^{+}_{k-q} \, a_k \, a_p$$

$$\llcorner \; U_q = U_0 \Rightarrow U(r) = U_0 \, \delta(r)$$

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$$U_q = U_0 \Rightarrow U(r) = U_0 \, \delta(r)$$

$$U_0 = \frac{4\pi \hbar^2}{m} a$$

a scattering length

$$a = \lim_{k \to 0} \left(- \frac{\delta_0}{k} \right) = -f$$

$$\hat{H}_1 = \frac{4\pi a \hbar^2}{m} \sum_{i<j} \delta(r_i - r_j) \cdot \underbrace{\frac{\partial}{\partial r_{ij}} r_{ij}}_{=1 \text{ in first order perturbation theory}}$$

Homogeneous interacting gas

$$H = \sum_k \varepsilon_k a_k^+ a_k + \frac{U_0}{2V} \sum_{k,k',q} a_{k+q}^+ a_{k'-q}^+ a_{k'} a_k$$

BEC in $k=0$ state

$$a_0^+ |N_0\rangle = \sqrt{N_0+1}\, |N_0+1\rangle$$

$$a_0 |N_0\rangle = \sqrt{N_0}\, |N_0-1\rangle$$

N_0 large $\quad N_0 \simeq N_0+1, \quad a_0 = a_0^+ = \sqrt{N_0}$

(Bogoliubov)

Homogeneous interacting gas

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(Bogoliubov)

$$H = \frac{U_0 N_0^2}{2V} + \sum_k \varepsilon_k a_k^\dagger a_k + \frac{U_0 N_0}{2V} \sum_{k \neq 0} a_k^\dagger a_{-k}^\dagger + a_k a_{-k} + 2 a_k^\dagger a_k + 2 a_{-k}^\dagger a_{-k}$$

Homogeneous interacting gas

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$$+ 2 a_k^+ a_k + 2 a_{-k}^+ a_{-k}$$

Introduce $N = N_0 + \frac{1}{2} \sum_{k\neq 0} a_k^+ a_k + a_{-k}^+ a_{-k}$

$$H = \frac{u_0 N^2}{2V} + \frac{1}{2}\sum_{k \neq 0} \left(\varepsilon_k + \frac{N u_0}{V} \right) \left(a_k^+ a_k + a_{-k}^+ a_{-k} \right) +$$

$$\frac{N u_0}{V} \left(a_k^+ a_{-k}^+ + a_k a_{-k} \right)$$

$$H = \frac{u_0 N^2}{2V} + \frac{1}{2}\sum_{k\neq 0}\left(\varepsilon_k + \frac{N u_0}{V}\right)\left(a_k^+ a_k + a_{-k}^+ a_{-k}\right) + \frac{N u_0}{V}\left(a_k^+ a_{-k}^+ + a_k a_{-k}\right)$$

Structure of H:

With $a = a_k$, $b = a_{-k}$

H has only terms of $\mathcal{H} = E_0(a^+a + b^+b) + E_1(a^+b^+ + ba)$

with $[a, a^+] = [b, b^+] = 1$

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bilinear expression solved by Bogoliubov transformation

$a = u\alpha - v\beta^+$ $\qquad$ $b = u\beta - v\alpha^+$

$u^2 - v^2 = 1$ ensures $[\alpha, \alpha^+] = [\beta, \beta^+] = 1$

canonical transformation

$$H = \frac{u_0 N^2}{2V} + \frac{1}{2}\sum_{k\neq 0}\left(\varepsilon_k + \frac{N u_0}{V}\right)\left(a_k^+ a_k + a_{-k}^+ a_{-k}\right) + \frac{N u_0}{V}\left(a_k^+ a_{-k}^+ + a_k a_{-k}\right)$$

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canonical transformation

$\mathcal{H} = (\ ) + (\ )(\alpha^+\alpha + \beta^+\beta) + (\ )(\alpha\beta + \beta^+\alpha^+)$

For choice of u, v $\quad = 0$

$$H = \lambda\left(\alpha^{+}\alpha + \beta^{+}\beta\right) + \text{const}$$

HO with quanta created by α^{+}, β^{+}

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HO with quanta created by α^{+}, β^{+}

Elementary excitations

$$H = \sum_{k} E_{k}\, a_{k}^{+} a_{k} + \text{Const}$$

$$E_{k} = \sqrt{\varepsilon_{k}^{2} + 2\,\varepsilon_{k} N u_{0}/V}$$

$$H = \lambda\left(\alpha^+\alpha + \beta^+\beta\right) + \text{Const}$$

HO with quanta created by α^+, β^+

Elementary excitations

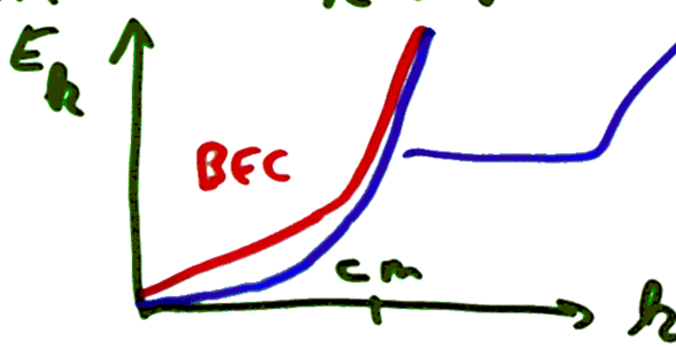
$$H = \sum_k E_k\, a_k^+ a_k + \text{Const}$$

$$E_k = \sqrt{\varepsilon_k^2 + 2\,\varepsilon_k \underbrace{N u_0/V}_{=c^2 m}}$$

$c = \sqrt{u_0 N/Vm}$ speed of sound

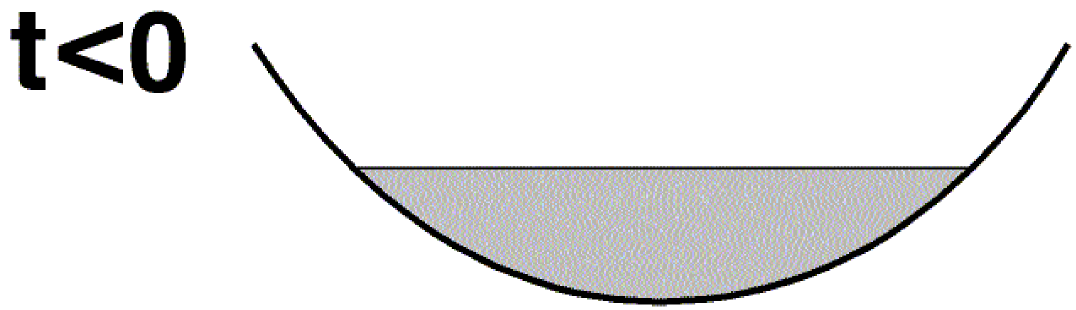
$$E_k = \sqrt{\left(\frac{\hbar^2 k^2}{2m}\right)^2 + \left(\hbar c k\right)^2}$$

$$= \begin{cases} \hbar c k & k \to 0 \quad \text{phonon, sound} \\ \hbar^2 k^2/2m & k \to \infty \quad \text{Free particle} \end{cases}$$

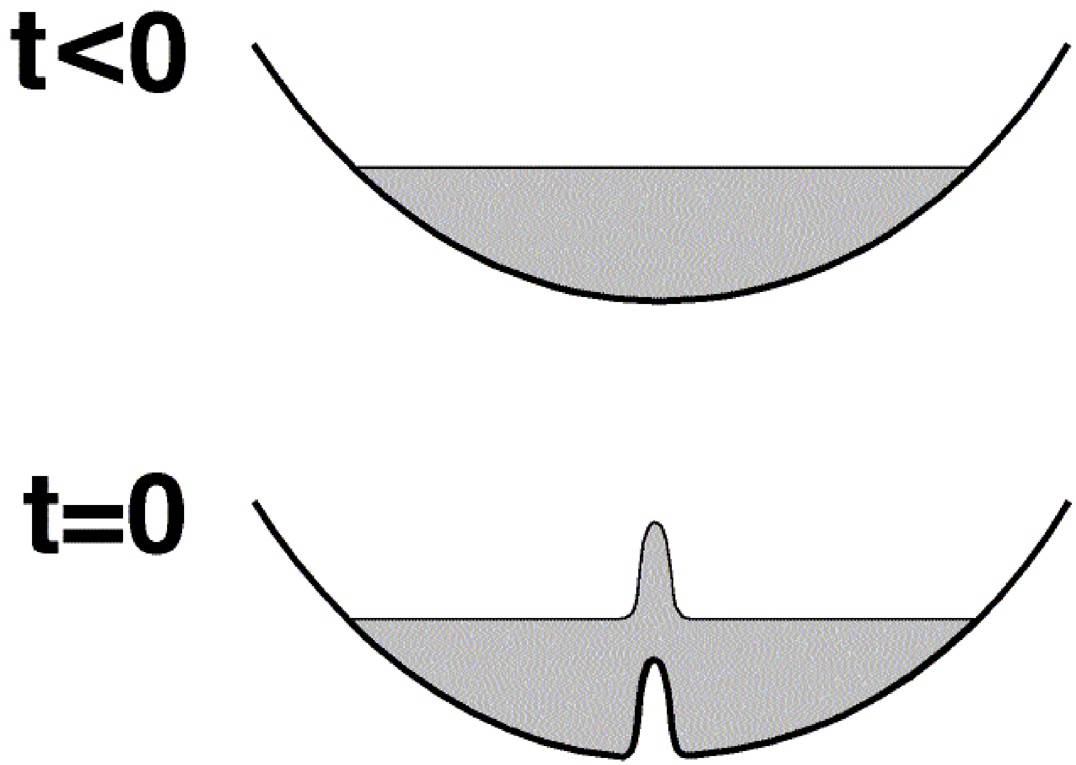


Propagation of sound

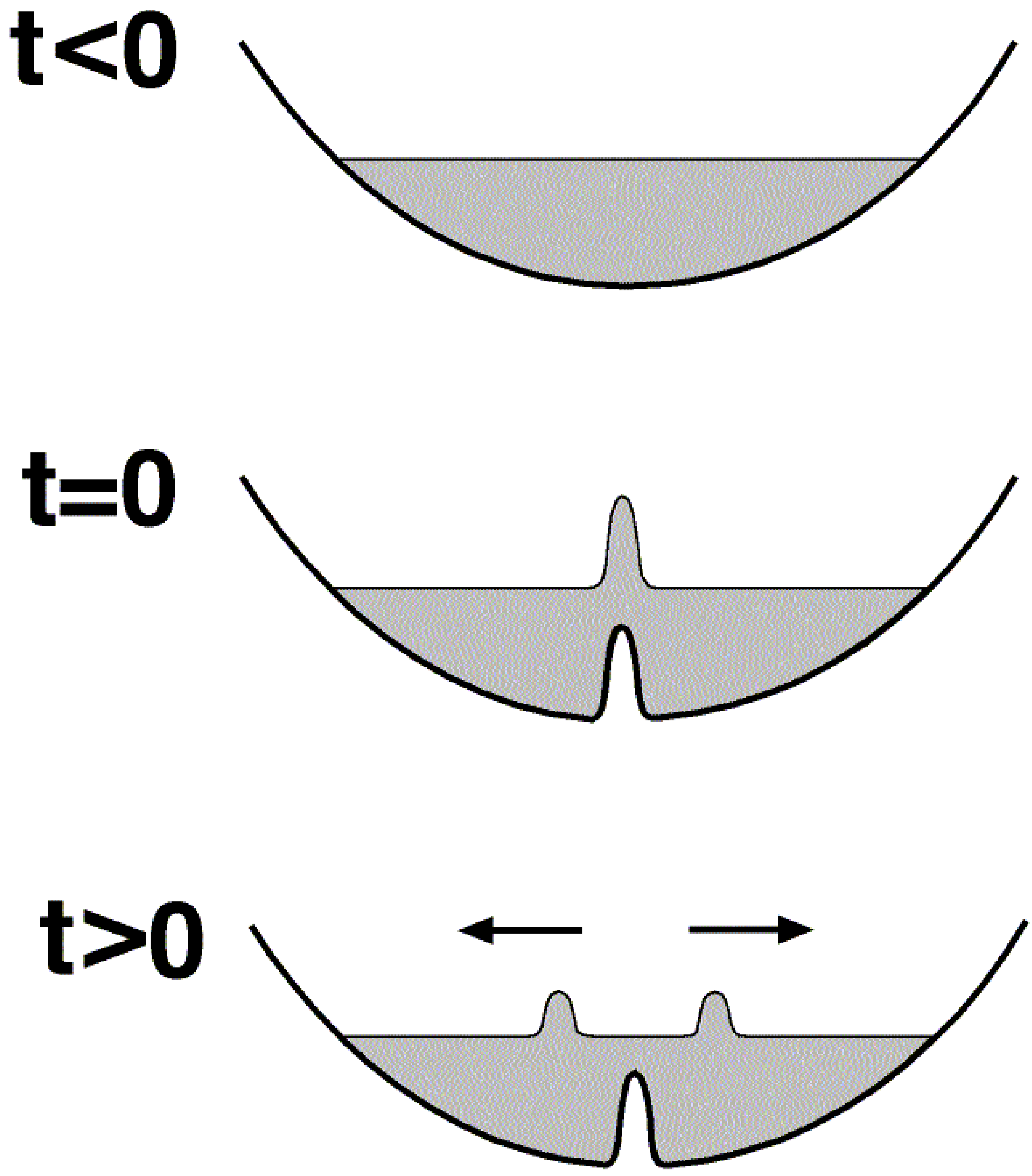
Excitation of sound



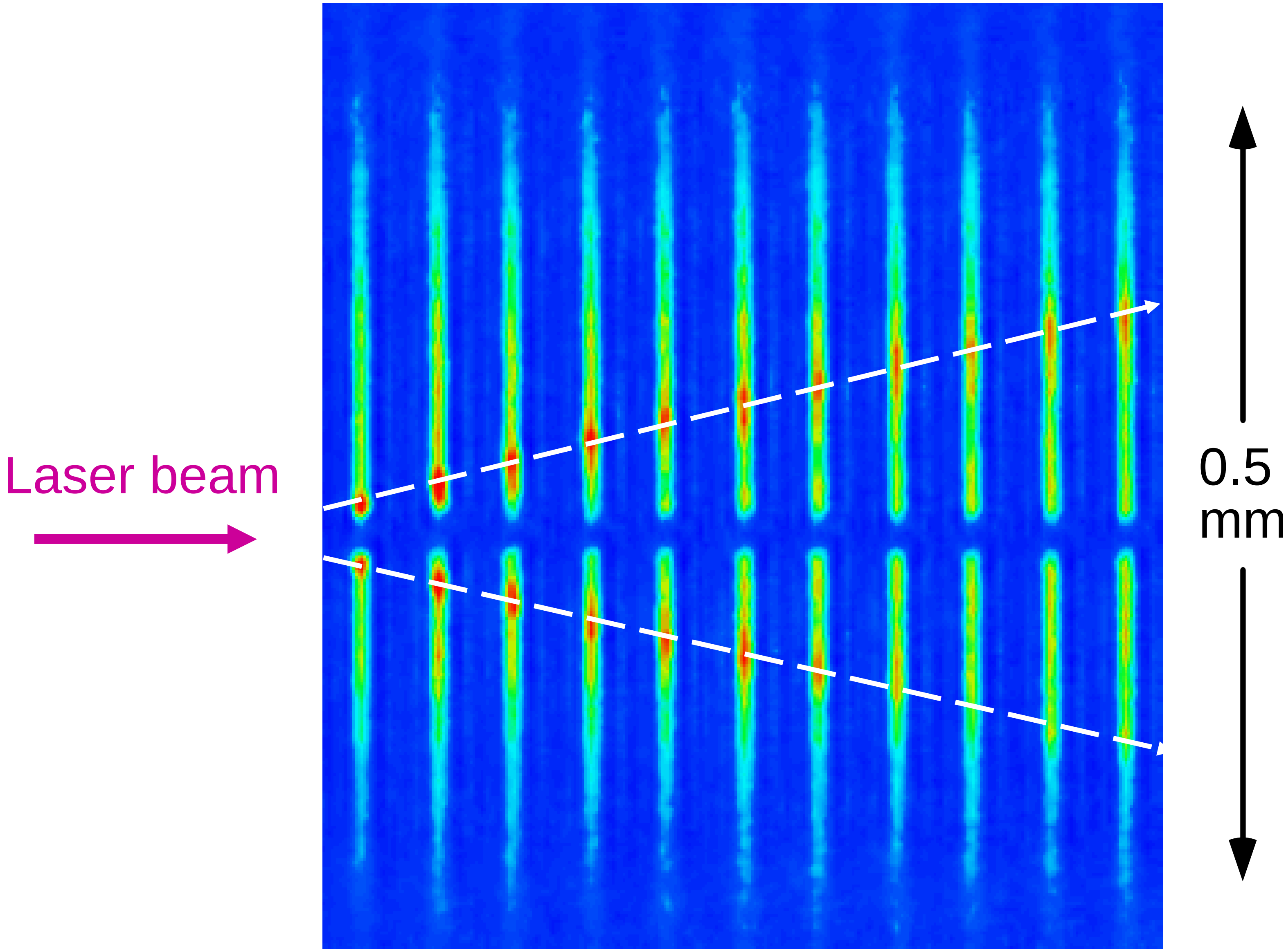
Excitation of sound



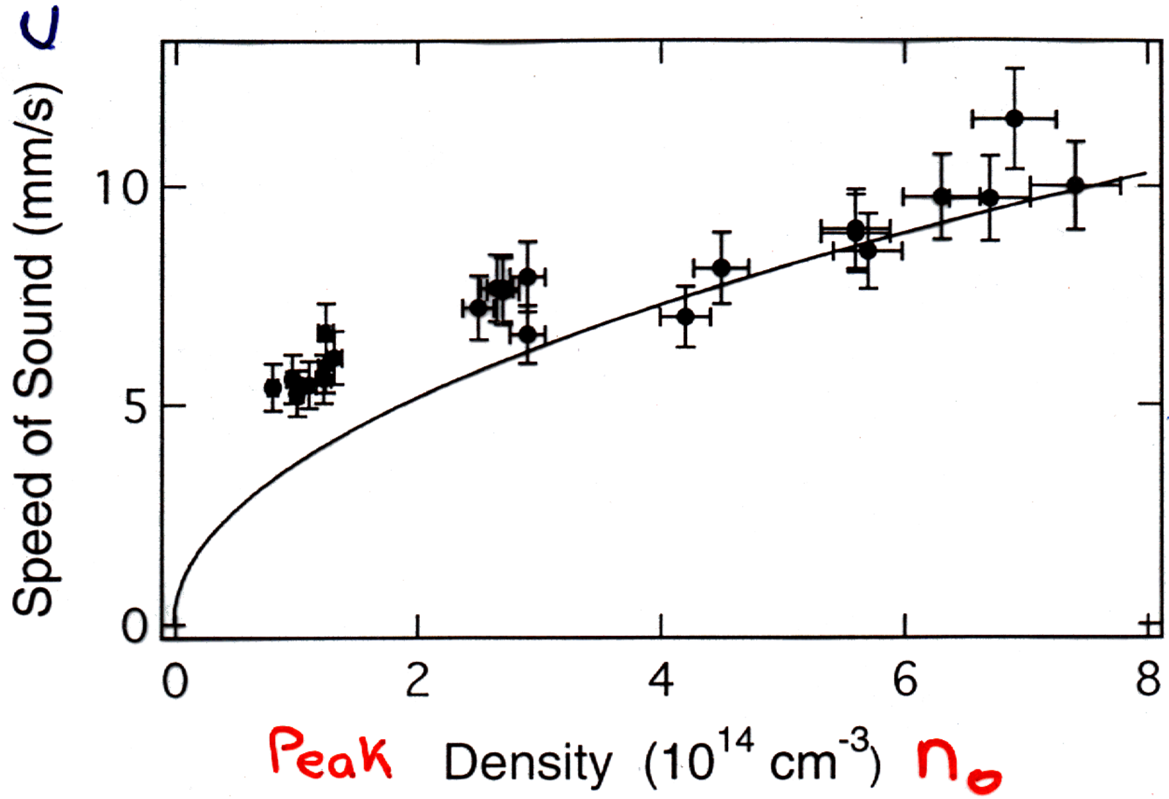
Excitation of sound



Sound = propagating density perturbations



1.3 ms per frame



$$c = \left(4\pi\hbar^2 a/m^2\right)^{1/2} \sqrt{n_o}/\sqrt{2}$$

Bogoliubov 1947

Lee, Huang, Yang 1957

(M. Andrews, D.M. Kurn, H.-J. Miesner, D.S. Durfee, C.G. Townsend, S. Inouye, W.K., PRL 79, 549 (1997))

Bogolinbov solution

$\rightarrow E_k$ elementary excitation

Bogoliubov solution

→ E_k elementary excitation

→ ground state energy

$$E_0 = \frac{U_0 n}{2}\left(1 + \frac{128}{15}\sqrt{na^3/\pi}\right)$$

Bogoliubov solution

→ E_k elementary excitation

→ ground state energy

$$E_0 = \frac{U_0 n}{2}\left(1 + \frac{128}{15}\sqrt{na^3/\pi}\right)$$

→ ground state wavefunction

$$\langle n_k \rangle = \frac{v_k^2}{1 - v_k^2}$$

$$\langle n_0 \rangle = N - \sum \langle n_k \rangle = N\Big[1 - \underbrace{\tfrac{8}{3}\sqrt{na^3/\pi}}_{\text{Quantum depletion}}\Big]$$

90% He
1% alkalis

$$|\psi_0\rangle = \left(|0\rangle\right)^N + \varepsilon$$

Quantum depletion
or
How to observe the transition from a quantum gas to a quantum liquid

In 1D: Zürich

K. Xu, Y. Liu, D.E. Miller, J.K. Chin, W. Setiawan, W.K., PRL 96, 180405 (2006).

What is the wavefunction of a condensate?

Ideal gas:

$$\Psi = \left(\left| q = 0 \right\rangle \right)^N$$

Interacting gas:

$$H' = U_0 \delta(r)$$

$$H' = U_0 \sum a_p^\dagger a_q^\dagger a_r a_s \qquad H' = U_0 a_0 a_0 \sum a_p^\dagger a_{-p}^\dagger$$

$$\Psi = \left(\left| q = 0 \right\rangle \right)^N + \alpha \left(\left| q = 0 \right\rangle \right)^{N-2} \left| q = p \right\rangle \left| q = -p \right\rangle + \ldots$$

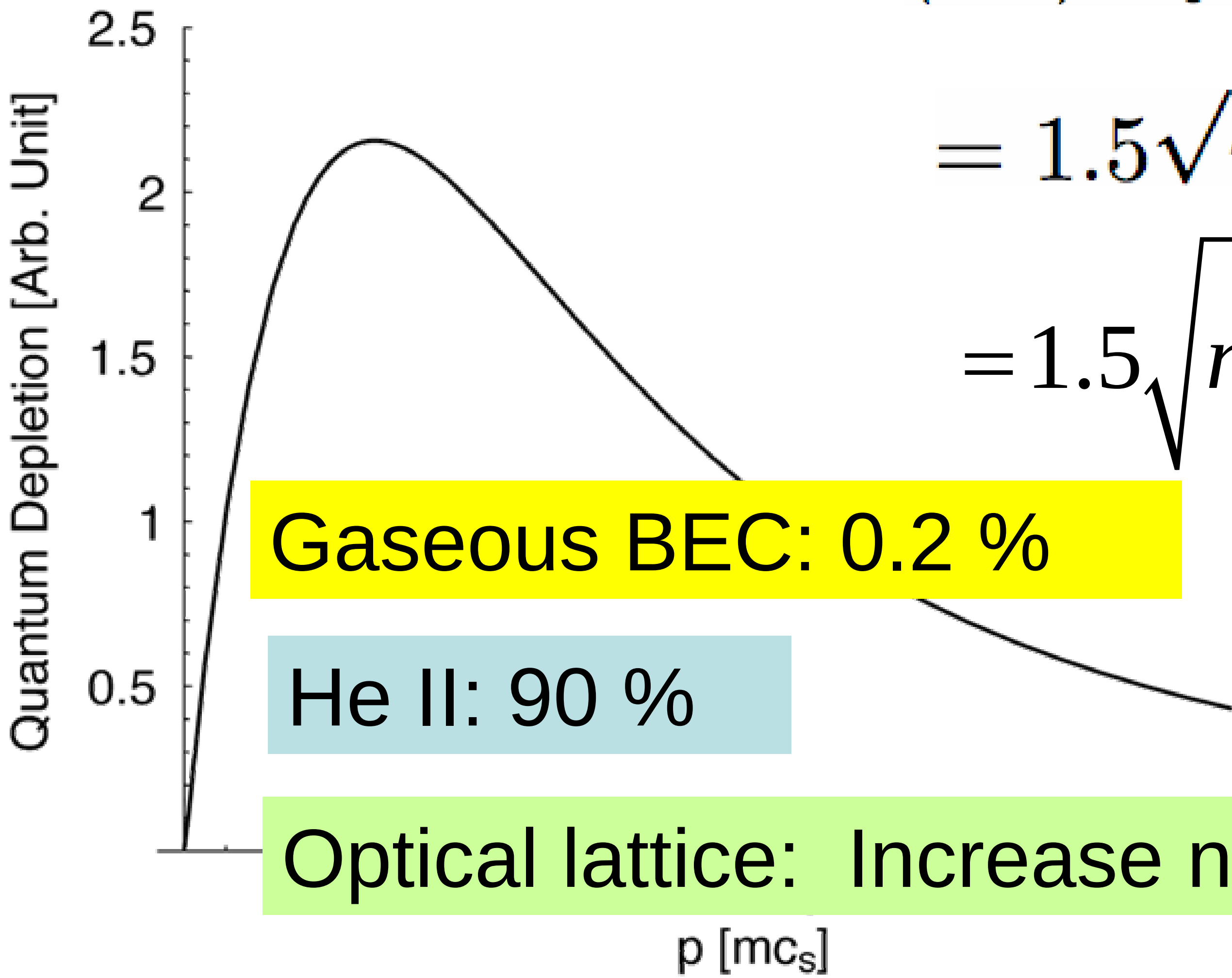
Quantum depletion

Quantum depletion in 3-dimensional free space

$$\frac{V}{(2\pi\hbar)^3}\int_0^\infty v_p^2(4\pi p^2)\,dp$$

$$= 1.5\sqrt{na^3}$$

$$= 1.5\sqrt{n\frac{M}{4\pi\hbar^2}U_0}$$



Gaseous BEC: 0.2 %

He II: 90 %

Optical lattice: Increase n and M_{eff}

Quantum Depletion

$$v_p^2 = \frac{T(p) + \mu - \sqrt{T^2(p) + 2\mu T(p)}}{2\sqrt{T^2(p) + 2\mu T(p)}}$$

Free space

$$T(p) = \frac{p^2}{2M}$$

$$\mu = \frac{4\pi\hbar^2 a}{M} n = M c_s^2$$

Lattice

$$4J \sin^2\left(\frac{\lambda_L}{4\hbar} p\right)$$

$$n_0 U$$

J : tunneling rate

U : on-site interaction

As one increases the depth of the optical lattice, the quantum depletion is dramatically increased

Finally, the condensate fraction becomes zero - a quantum phase transition to an insulator is taking place.

Dispersion relation

Elementary excitations

$$H = \sum_k E_k \, a_k^+ a_k + \text{const}$$

$$E_k = \sqrt{\varepsilon_k^2 + 2\,\varepsilon_k \underbrace{N u_0 / V}_{= c^2 m}}$$

$c = \sqrt{u_0 N / V m}$ speed of sound

$$E_k = \sqrt{\left(\frac{\hbar^2 k^2}{2m}\right)^2 + (\hbar c k)^2}$$

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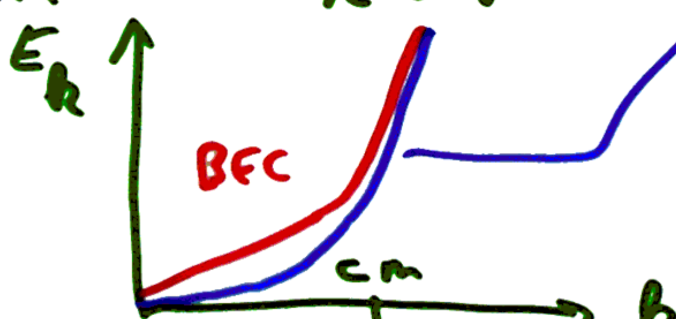
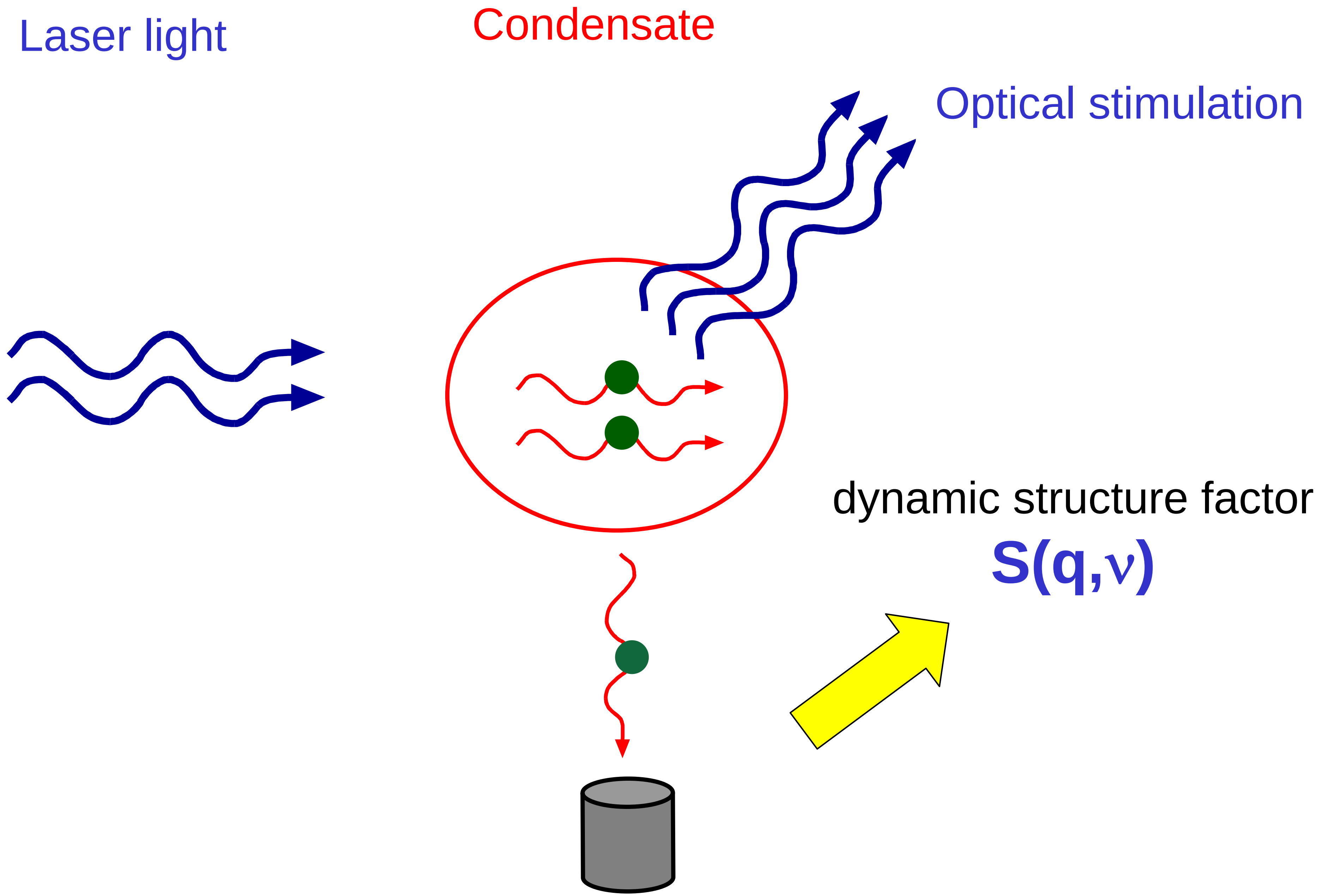
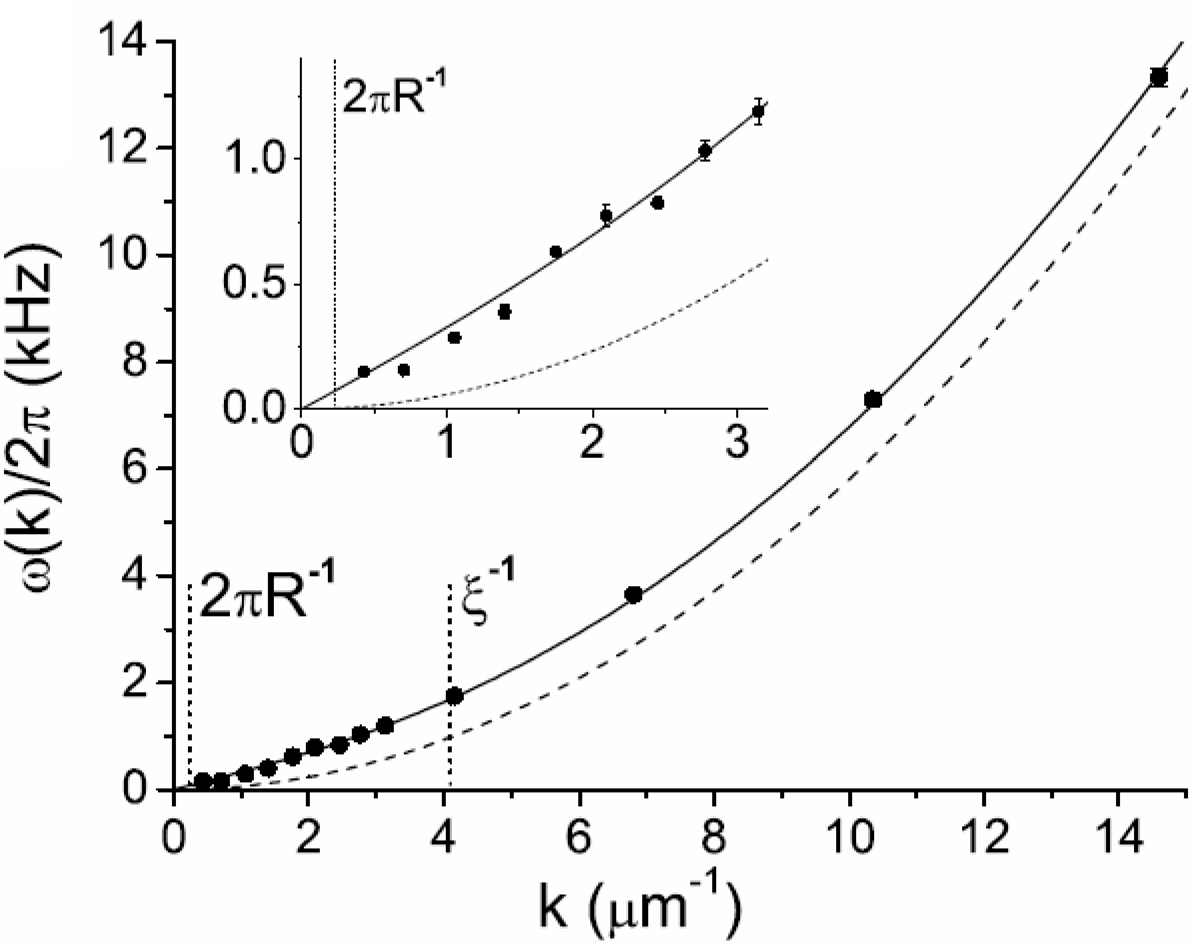



Laser light
Condensate
Optical stimulation
dynamic structure factor
S(q,ν)

Excitation Spectrum of a Bose-Einstein Condensate

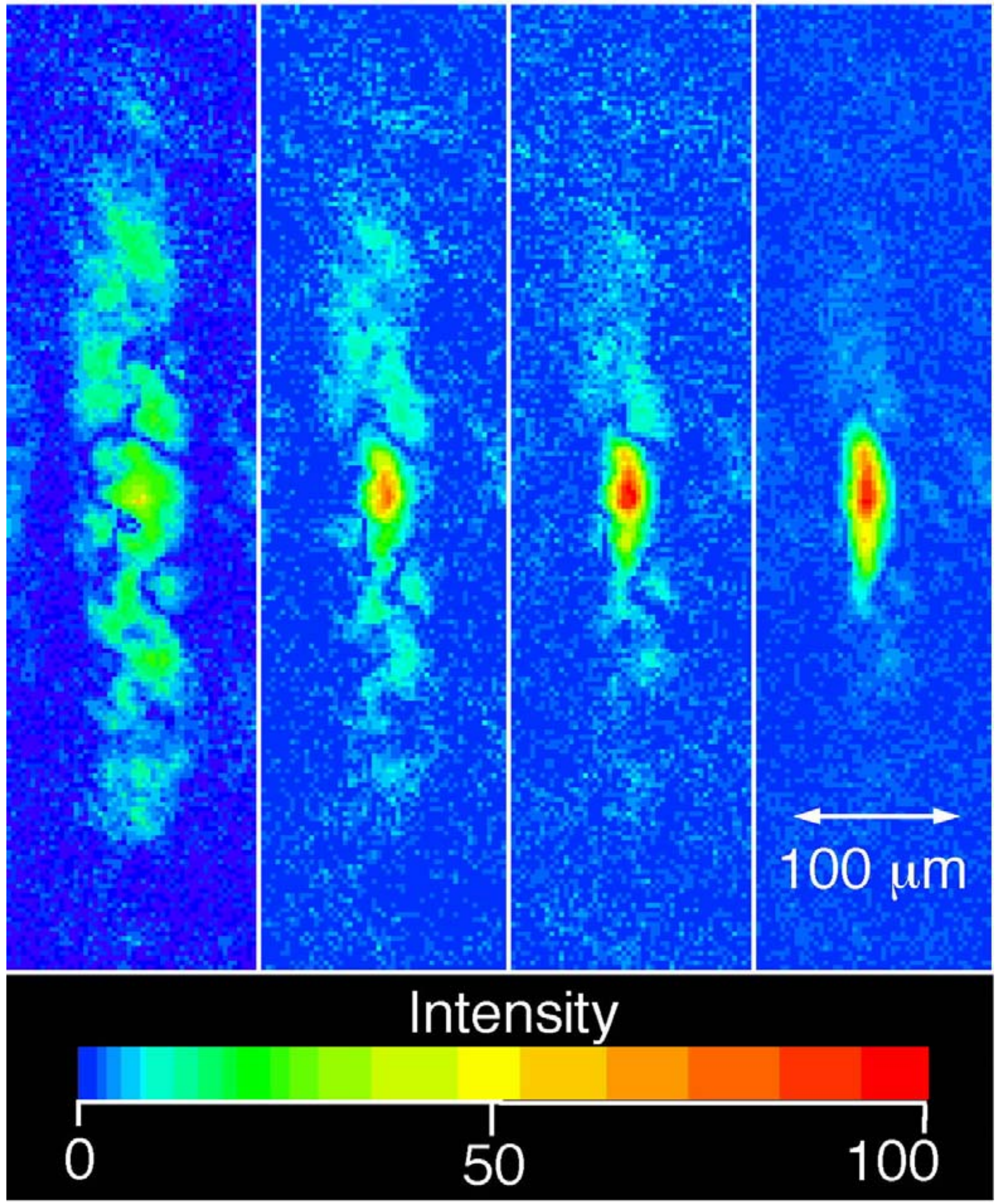
J. Steinhauer, R. Ozeri, N. Katz, and N. Davidson



Inhomogeneous BEC

A live condensate in the magnetic trap

(seen by dark-ground imaging)



The inhomogeneous Bose gas

New Feature: trapping potential

$\rightarrow$ go to $\vec{r}$ space

$$\hat{H} = \int d^3r\, \hat{\psi}^+(r) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{trap} \right] \hat{\psi}(r)$$

$$+ \frac{1}{2} \int d^3r \int d^3r'\, \hat{\psi}^+(r) \hat{\psi}^+(r')\, U(r-r')\, \hat{\psi}(r') \hat{\psi}(r)$$

The inhomogeneous Bose gas

New feature: trapping potential

$\rightarrow$ go to $\vec{r}$ space

$$\underbrace{\hat{H} = \int d^3r \, \hat{\psi}^+(r) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{trap} \right] \hat{\psi}(r) + \frac{1}{2} \int d^3r \int d^3r' \, \hat{\psi}^+(r) \hat{\psi}^+(r') \underbrace{U(r-r')}_{U_0 \delta(r-r')} \hat{\psi}(r') \hat{\psi}(r)}_{\frac{U_0}{2} \int d^3r \, \hat{\psi}^+(r) \hat{\psi}^+(r) \hat{\psi}(r) \hat{\psi}(r)}$$

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$$\underbrace{+ \frac{1}{2} \int d^3r \int d^3r' \; \hat{\psi}^+(r) \hat{\psi}^+(r') \underbrace{U(r-r')}_{U_0 \delta(r-r')} \hat{\psi}(r') \hat{\psi}(r)}_{\frac{U_0}{2} \int d^3r \; \hat{\psi}^+(r) \hat{\psi}^+(r) \hat{\psi}(r) \hat{\psi}(r)}$$

Heisenberg equation of motion for $\hat{\psi}$

$$i\hbar \frac{\partial \hat{\psi}(r,t)}{\partial t} = [\hat{\psi}, \hat{H}]$$ cannot be solved

Bogoliubov: condensate operator → c-number

$$\hat{\psi}(r,t) = \psi(r,t) + \tilde{\psi}(r,t)$$

$\psi(r,t)$: $\langle \hat{\psi}(r,t) \rangle$

$\tilde{\psi}(r,t)$: quantum (+ thermal) fluctuations

Bogoliubov: condensate operator $\rightarrow$ c-number

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$\psi(r,t)$: $\langle \hat{\psi}(r,t) \rangle$

$\tilde{\psi}(r,t)$: quantum (+ thermal) fluctuations

$T \rightarrow 0$ neglect $\tilde{\psi}$

$\rightarrow$ NLSE, Gross-Pitaevskii equation for $\psi(r,t)$

$$i\hbar \frac{\partial \psi}{\partial t} = \Big[-\frac{\hbar^2}{2m}\nabla^2 + V_{trap} + \underbrace{u_0 \underbrace{N|\psi(r,t)|^2}_{n(r,t)\ \text{density}}}_{\text{mean field potential}}\Big]\psi(r,t)$$

$$u_0 \sum \delta(r) \rightarrow u_0 n(r,t)$$

Thomas-Fermi approximation

ξ healing length

$$\frac{\hbar^2}{2m\xi^2} = U_0 n$$

$$\xi = (8\pi a n)^{-1/2}$$

Thomas-Fermi approximation

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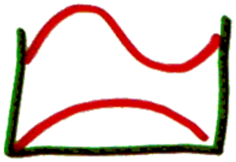
neglect kinetic energy term

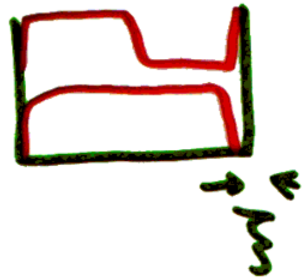
$$\left[\underbrace{\cancel{-\frac{\hbar^2}{2m}\nabla^2}}_{\to 0} + V_{ext} + U_0 |\psi(r)|^2\right]\psi(r) = \mu\psi(r)$$

$$\left[V_{ext} + U_0 |\psi(r)|^2 - \mu\right]\psi(r) = 0$$

Thomas-Fermi approximation



$\xrightarrow[\text{actions}]{}$ inter-actions



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$$\frac{\hbar^2}{2m\xi^2} = U_0 n$$

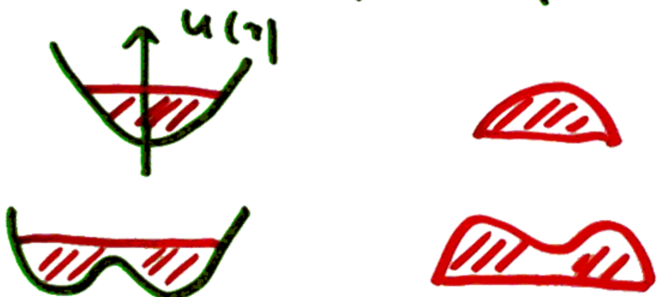
$$\xi = (8\pi a n)^{-1/2}$$

neglect kinetic energy term

$$\left[\cancel{-\frac{\hbar^2}{2m}\nabla^2}^{\,\to 0} + V_{ext} + U_0 |\psi(r)|^2 \right] \psi(r) = \mu \psi(r)$$

$$\left[V_{ext} + U_0 |\psi(r)|^2 - \mu \right] \psi(r) = 0$$

Solution: $|\psi(r)|^2 = \frac{1}{U_0}\left(\mu - V(r)\right)$



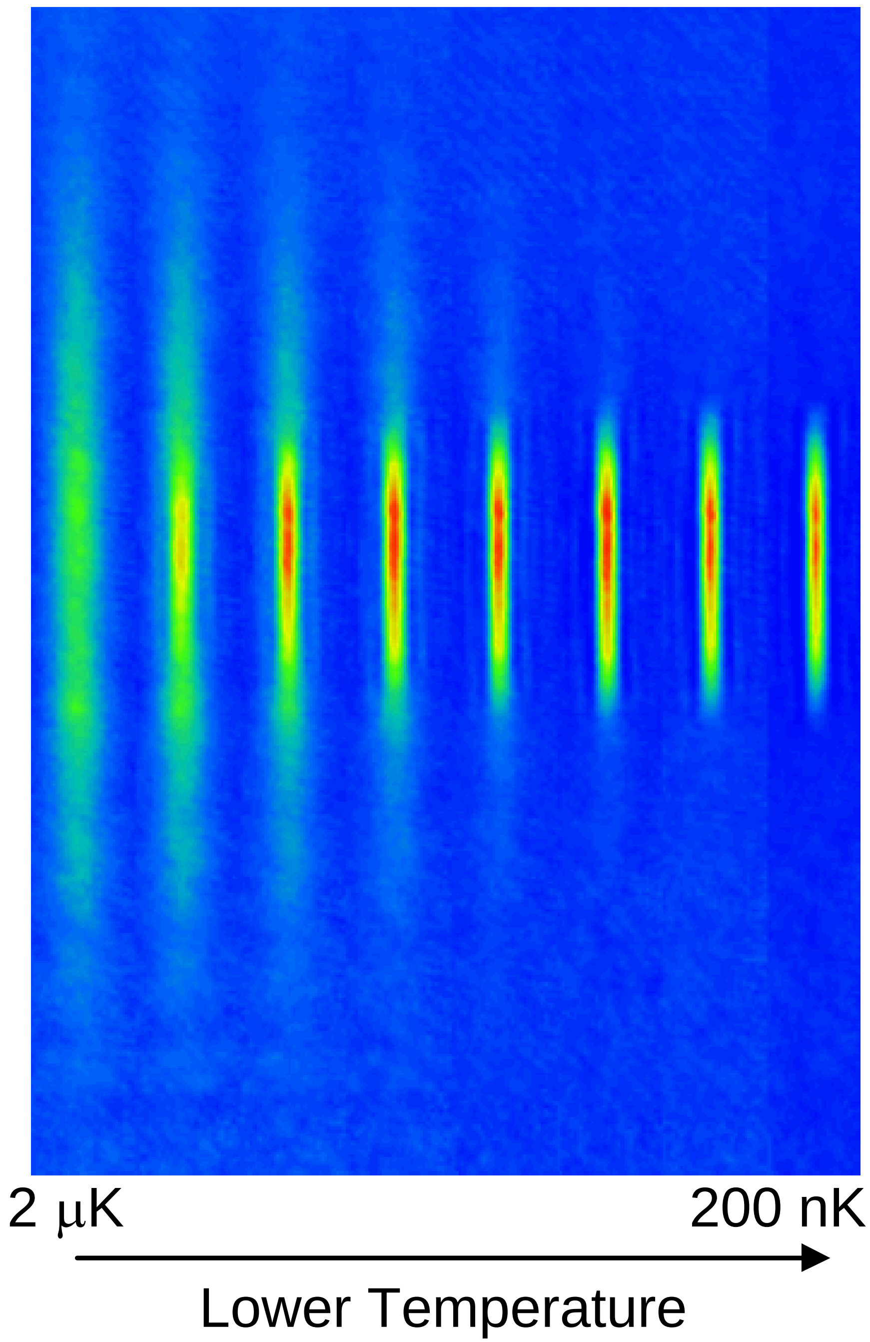
Shape is inverted trapping potential

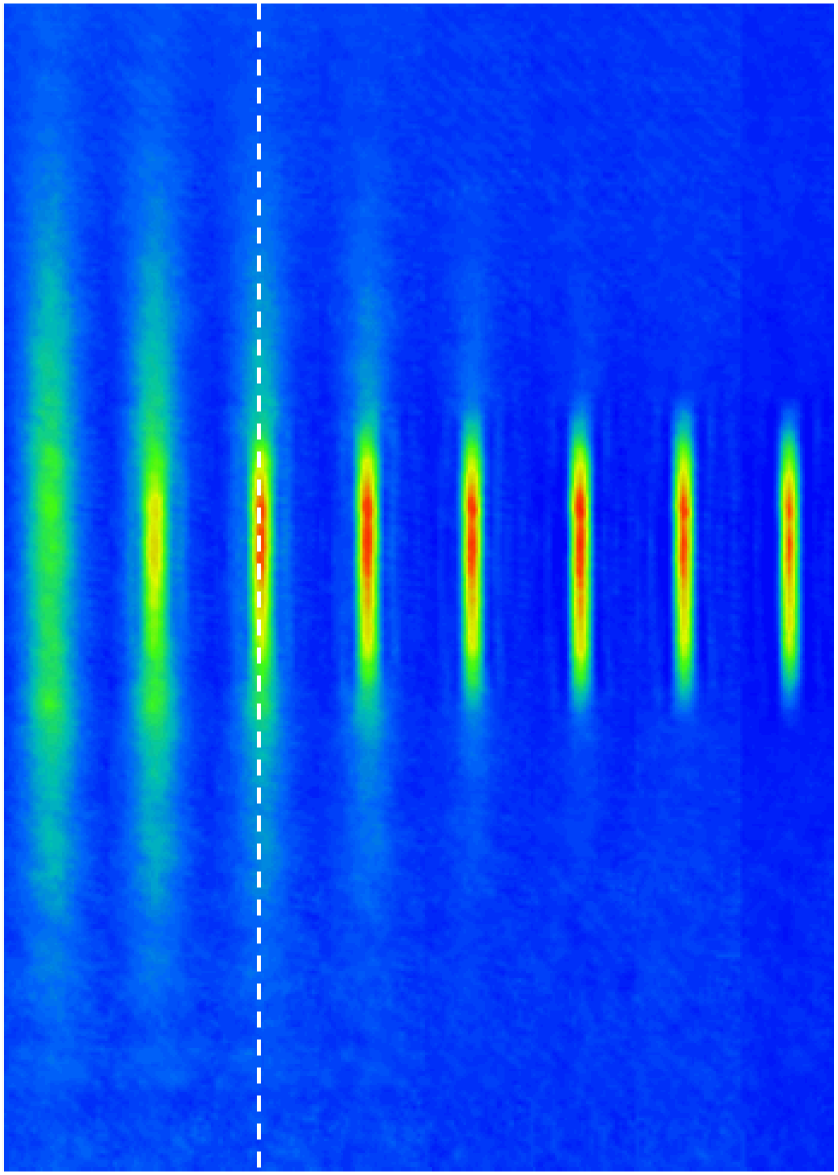
HO: $V(r) = \frac{1}{2} m \omega^2 r^2$

$$\mu = \frac{\hbar\omega}{2} \underbrace{\left(\frac{15 N a}{a_{osc}}\right)}_{x}{}^{2/5} \qquad a_{osc} = \sqrt{\frac{\hbar}{m\omega}}$$

$$\mu = x^{2/5} \frac{\hbar\omega}{2} = \frac{1}{2} m \omega^2 R^2$$

$$R = a_{osc}\, x^{1/5}$$


2 μK
200 nK
Lower Temperature



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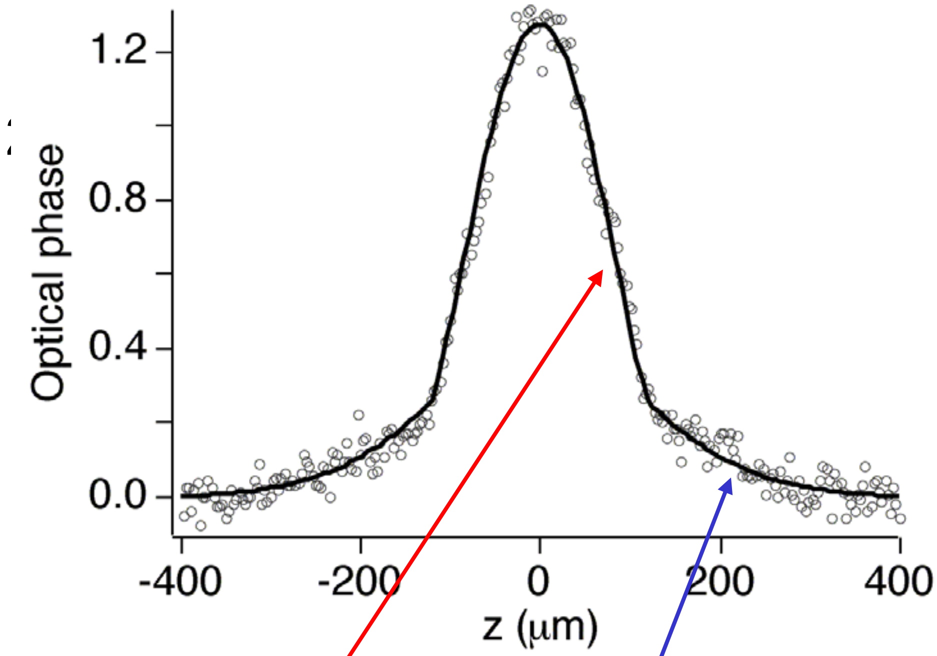
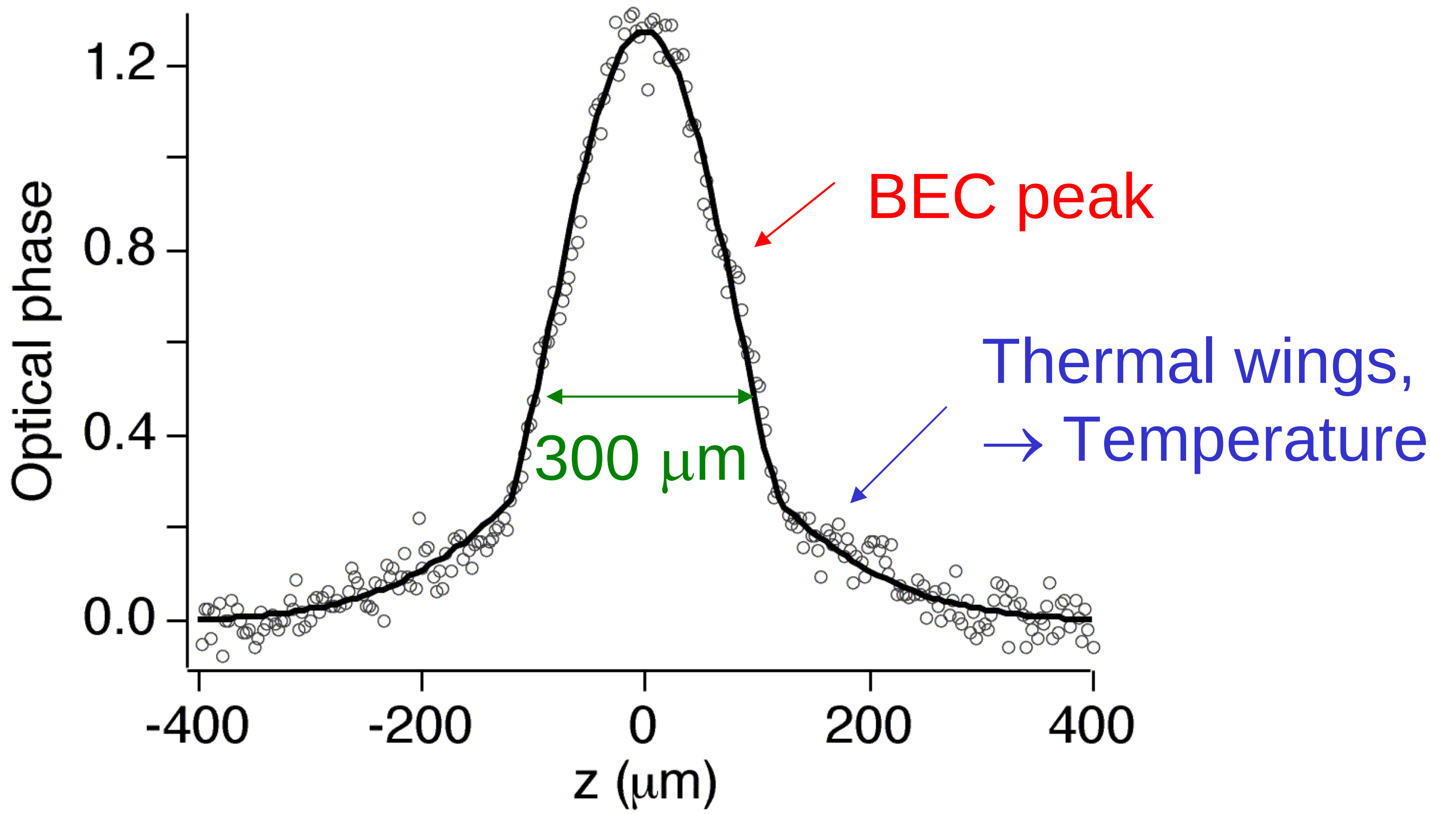
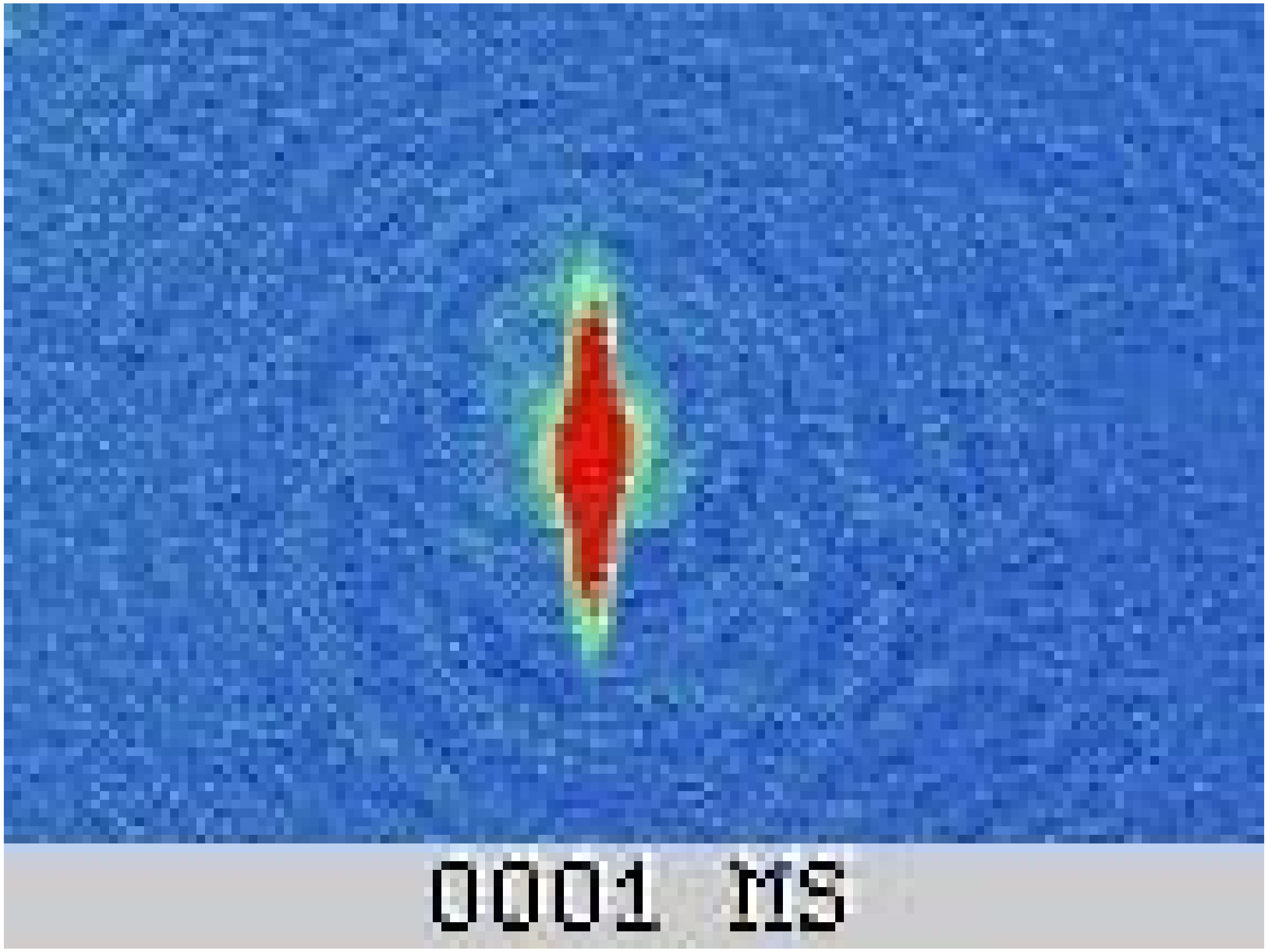




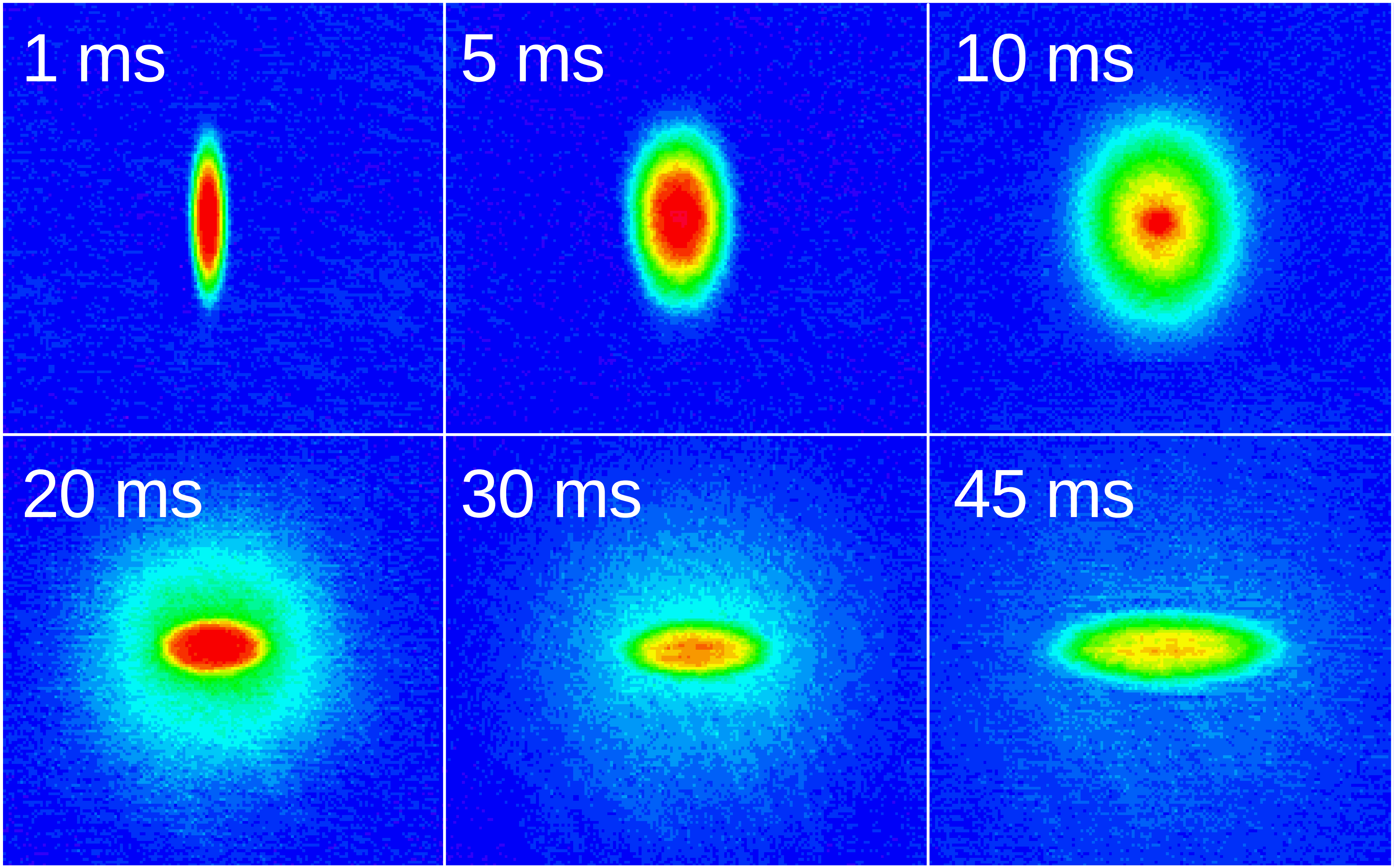
rms width of harmonic oscillator ground state 7 μm

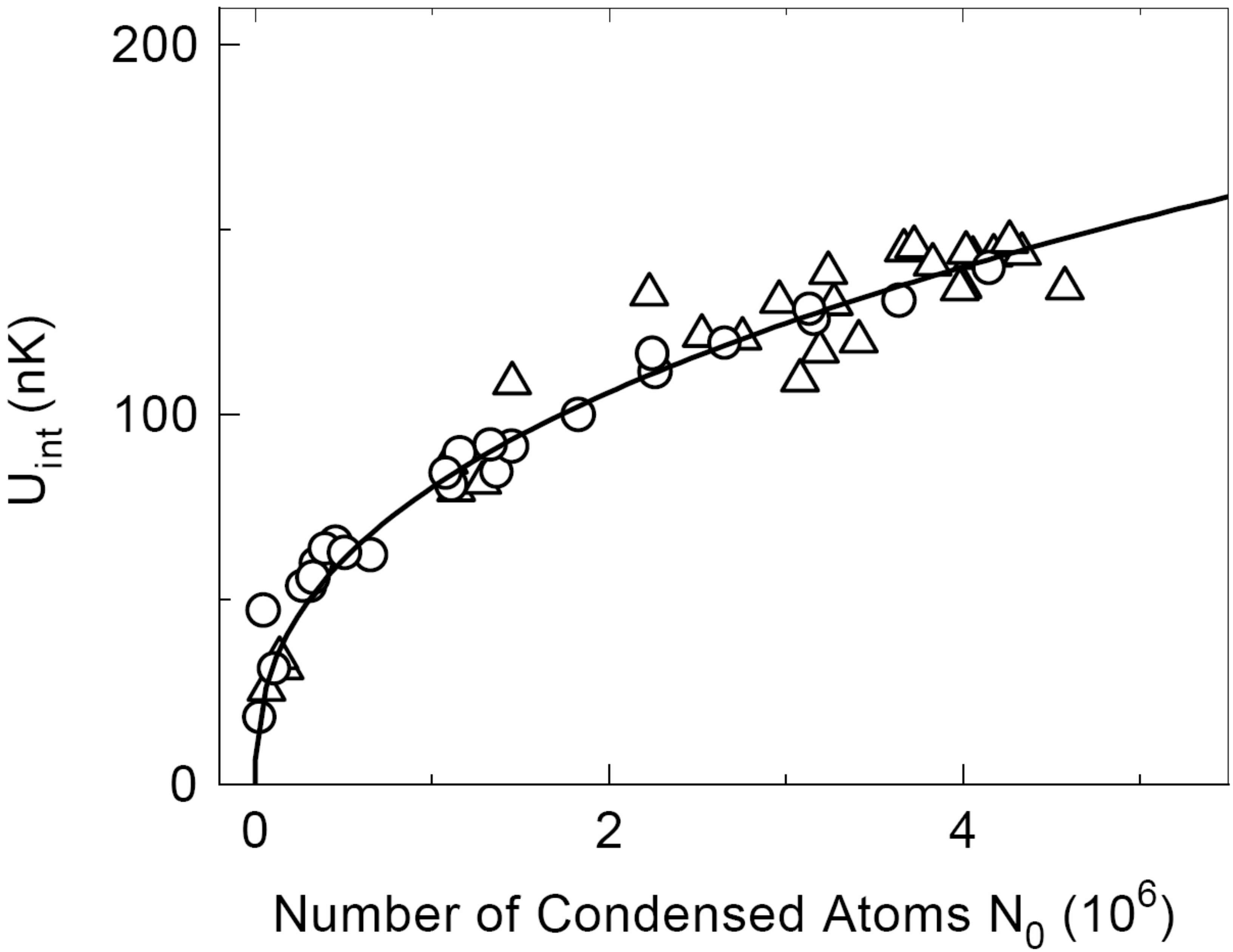
⇒ (repulsive) interactions

⇒ interesting many-body physics


0001 MS

Signatures of BEC: Anisotropic expansion




200
100
0
0
2
4
U_{int} (nK)
Number of Condensed Atoms N_0 (10^6)

Vortices

→ NLSE, Gross-Pitaevskii equation for $\psi(r,t)$

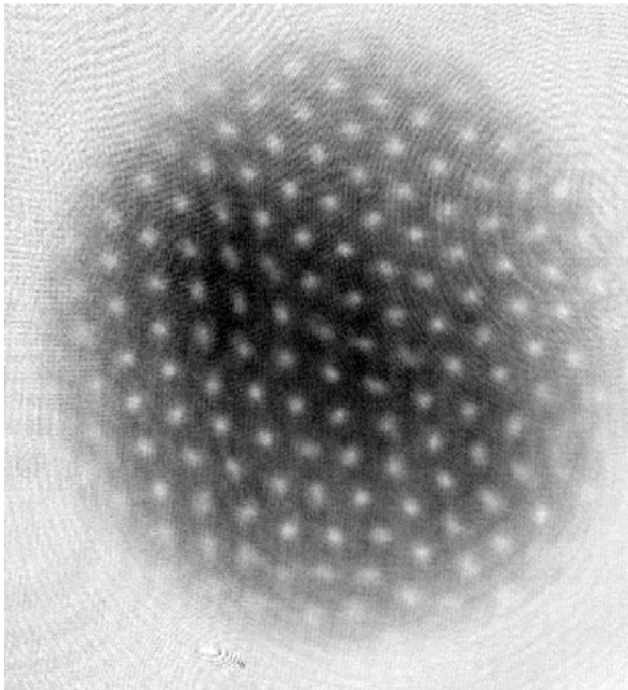
$$i\hbar \frac{\partial \psi}{\partial t} = \Big[-\frac{\hbar^2}{2m}\nabla^2 + V_{trap} + \underbrace{U_0 \underbrace{N|\psi(r,t)|^2}_{n(r,t)\ \text{density}}}_{\text{mean field potential}} \Big] \psi(r,t)$$

$$U_0 \sum \delta(r) \longrightarrow U_0 n(r,t)$$

Spinning a Bose-Einstein condensate

The rotating bucket experiment with a superfluid gas 100,000 thinner than air

Rotating
green laser beams



Two-component vortex
- Boulder, 1999

Single-component vortices
- Paris, 1999
- Boulder, 2000
- MIT 2001
- Oxford 2001

J. Abo-Shaeer, C. Raman, J.M. Vogels, W.Ketterle, Science, 4/20/2001

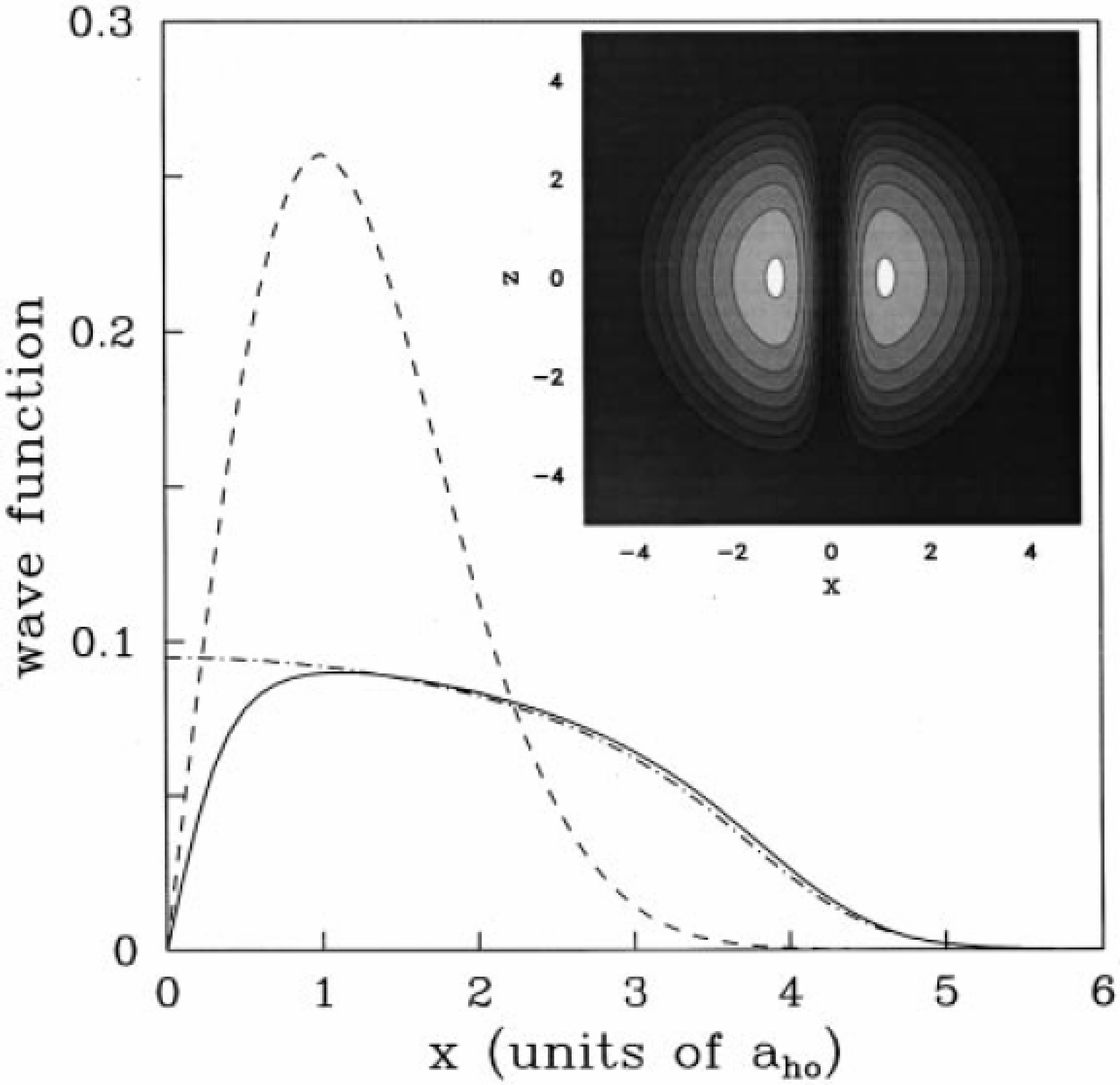
GPE for vortices

Order parameter

$$\phi(\mathbf{r}) = \phi_v(r_\perp, z)\exp[i\kappa\varphi]$$

GPE for modulus

$$\left[-\frac{\hbar^2\nabla^2}{2m} + \frac{\hbar^2\kappa^2}{2mr_\perp^2} + \frac{m}{2}(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2) + g\phi_v^2(r_\perp, z)\right]\phi_v(r_\perp, z) = \mu\phi_v(r_\perp, z)$$



F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999)

Hydrodynamics

Hydrodynamic Flow of a superfluid

15

GPE
$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r) + U_0|\psi|^2\right)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$* \ \psi^*$, subtract c.c.

$$\frac{\partial|\psi|^2}{\partial t} + \nabla\frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) = 0$$

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$$\frac{\partial n}{\partial t}$$

$$v = \frac{j}{n} = \frac{\psi^*\nabla\psi - \psi\nabla\psi^*}{2mi|\psi|^2}$$

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$$\frac{\partial n}{\partial t} + \nabla(nv) = 0 \qquad \text{Continuity equation}$$

$$\psi = f e^{i\phi} \Rightarrow v = \frac{\hbar}{m}\nabla\phi$$

↳ Quant. of circulation $\oint \vec{v}\, d\vec{s}$

vortices

persistent currents

macroscopic wavefunction "→" SF flow

insert $\psi = f e^{i\phi}$ into NLSE, separate real and im. parts

$$-\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2mf} \nabla^2 f + \frac{1}{2} m v^2 + V(r) + u_0 f^2$$

take ∇

$$m \frac{\partial v}{\partial t} = -\nabla \left(\delta\mu + \frac{1}{2} m v^2 \right)$$

$$\delta\mu = V + u_0 n - \frac{\hbar^2}{2m\sqrt{n}} \nabla^2 \sqrt{n} \underbrace{- \mu_0}_{\text{arb. const}}$$

exact! $\quad \delta\mu = 0 \;\hat{=}\;$ time indep. GPE equation insert $\psi = f e^{i\phi}$ into NLSE, separate real and im. parts

$$-\hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2mf} \nabla^2 f + \frac{1}{2} m v^2 + V(r) + u_0 f^2$$

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exact ! $\delta\mu = 0 \;\hat{=}\;$ time indep. GPE equation

Now: Thomas Fermi approx.

neglect ∇f (density derivative)

but not $\nabla \phi$

$n = n_0 + \delta n$ where $n_0 u_0 + V = \mu_0$

$\delta\mu = u_0 \, \delta n$

eliminate $\delta\mu$, neglect v^2 (higher order)

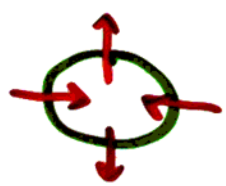
$$m \frac{\partial v}{\partial t} = -\nabla (u_0 \cdot \delta n) \qquad \Big| * n_0 ,\ \nabla$$

$$\left.\begin{array}{l} m\nabla\left(n_0 \dfrac{\partial v}{\partial t}\right) = -U_0 \nabla\left(n_0 \left(\nabla \delta n\right)\right) \\[2ex] \dfrac{\partial \delta n}{\partial t} + \nabla(n_0 v) = 0 \qquad \text{linearized} \end{array}\right\} \text{Combine}$$

$$m \frac{\partial^2 \delta n}{\partial t^2} = U_0 \nabla(n_0 \nabla \delta n)$$

- For $n_0 = \text{const}$ Wave equation for δn
  velocity $c = \sqrt{U_0/m}$ Bogoliubov

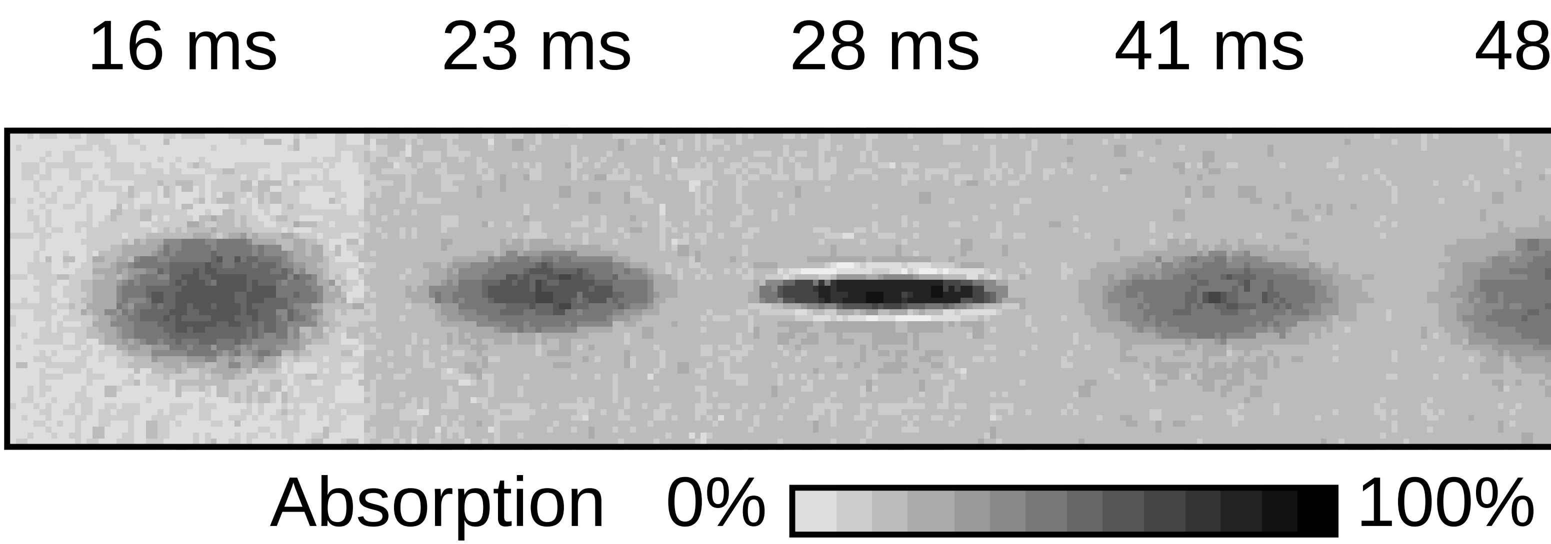
- TF solution for n_0
  $\Rightarrow$ discrete modes, $e^{\pm i m \phi}$ m angular mom
  e.g. $\omega = \sqrt{2}\, \omega_{trap}$



Shape oscillations of the cloud

Collective excitations

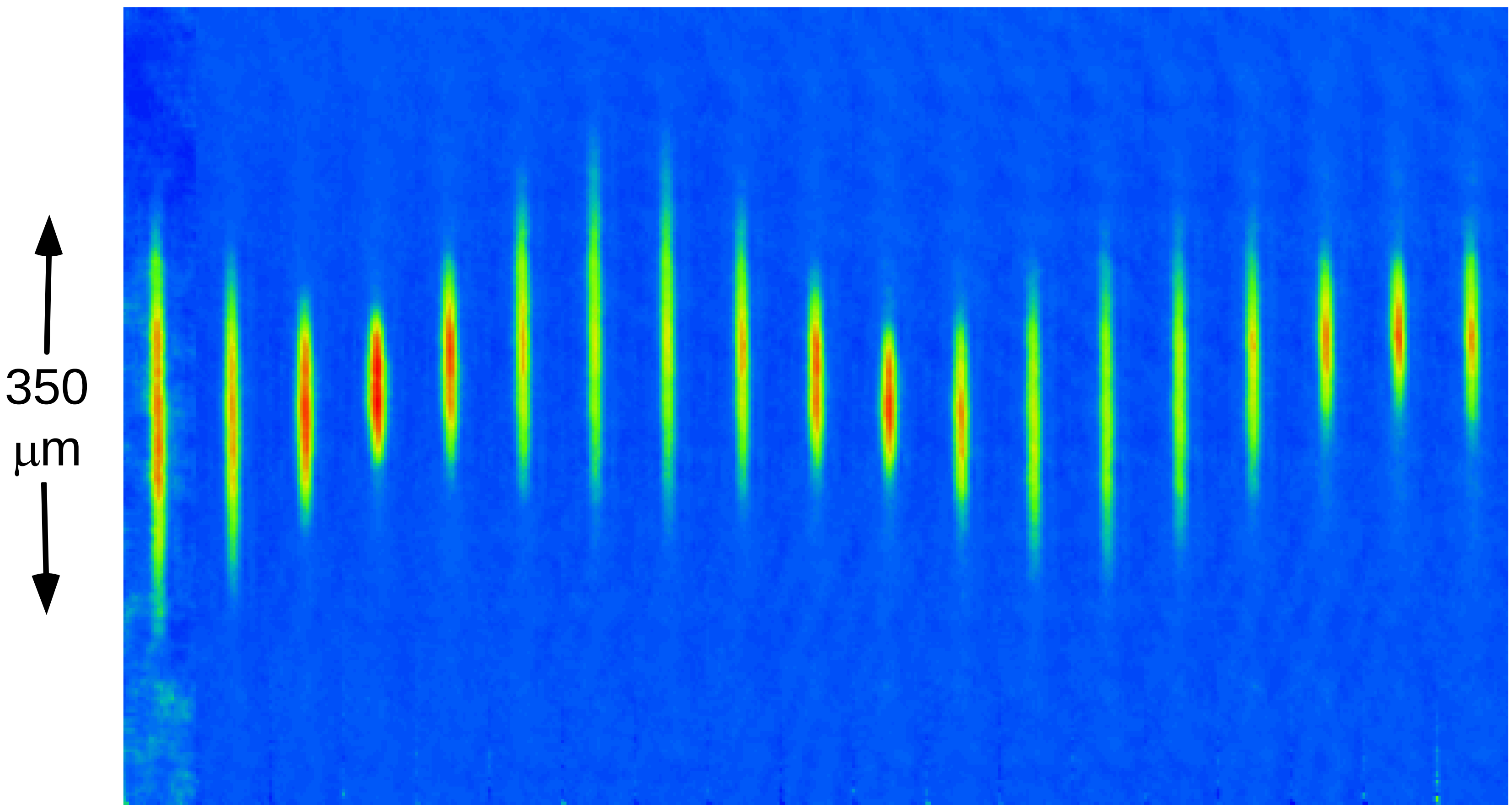
(observed in ballistic expansion)



MIT, 1996

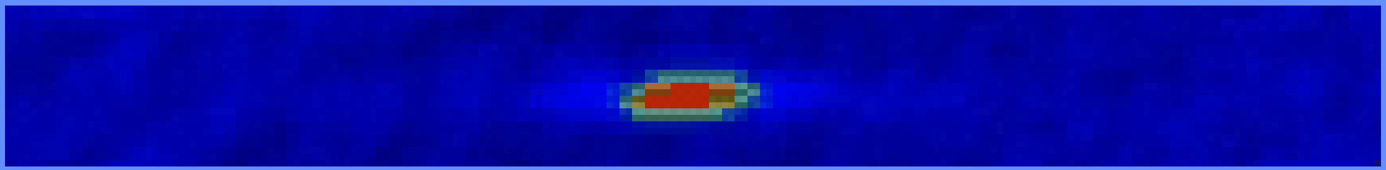
Shape oscillations

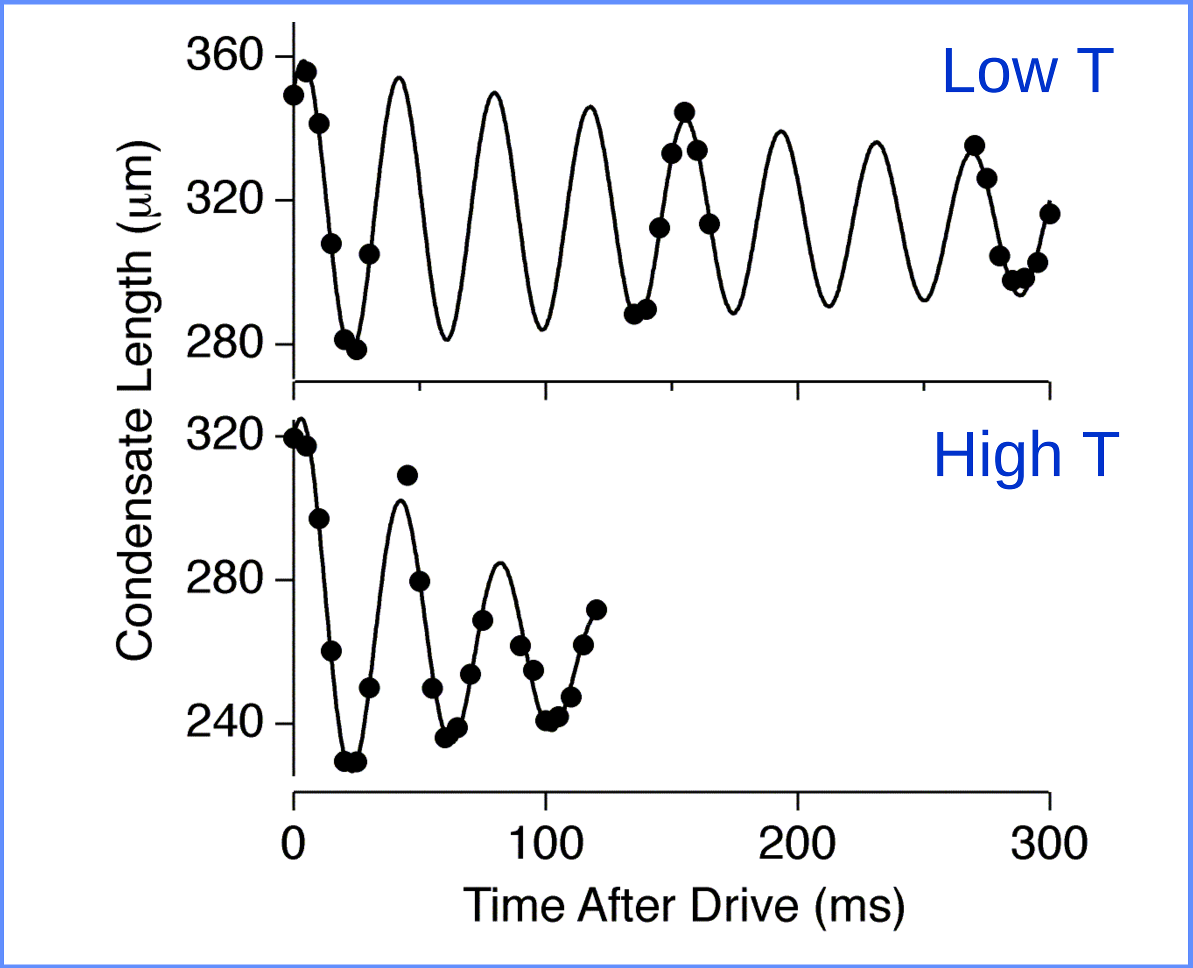
"Non-destructive" observation of a time-dependent wave function



5 milliseconds per frame

m=0 quadrupole-type oscillation at 29 Hz





Stamper-Kurn, Miesner, Inouye, Andrews, W.K, PRL **81**, 500 (1998)

Optical Lattices

Superfluid to Mott Insulator Transition

Optical lattice (cubic)

$$V(x,y,z) = V_0 \left(\sin^2 kx + \sin^2 ky + \sin^2 kz \right)$$

QM in periodic potentials (1D)

$$H = \frac{\hbar^2}{2m} \nabla^2 - V_0 \sin^2(kx)$$

$$\psi_{q,n} = e^{iqx/\hbar} u_{q,n}(x) \qquad \text{Bloch theorem}$$

Quasi momentum (q)

Band index (n)

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q: Quasi momentum, n: Band index

Fourier expansion

$$-V_0 \sin^2(kx) = \sum \tilde{V}_r e^{i2rkx} \qquad \Bigg| \qquad \sum_{\ell} c_{\ell}^{q,n} e^{i2\ell kx}$$

$$\tilde{V}_{-1} = \tilde{V}_1 = V_0/4$$

$$\tilde{V}_0 = -V_0/2$$

Insert into $H\psi_{q,n} = E_{q,n}\psi_{q,n}$

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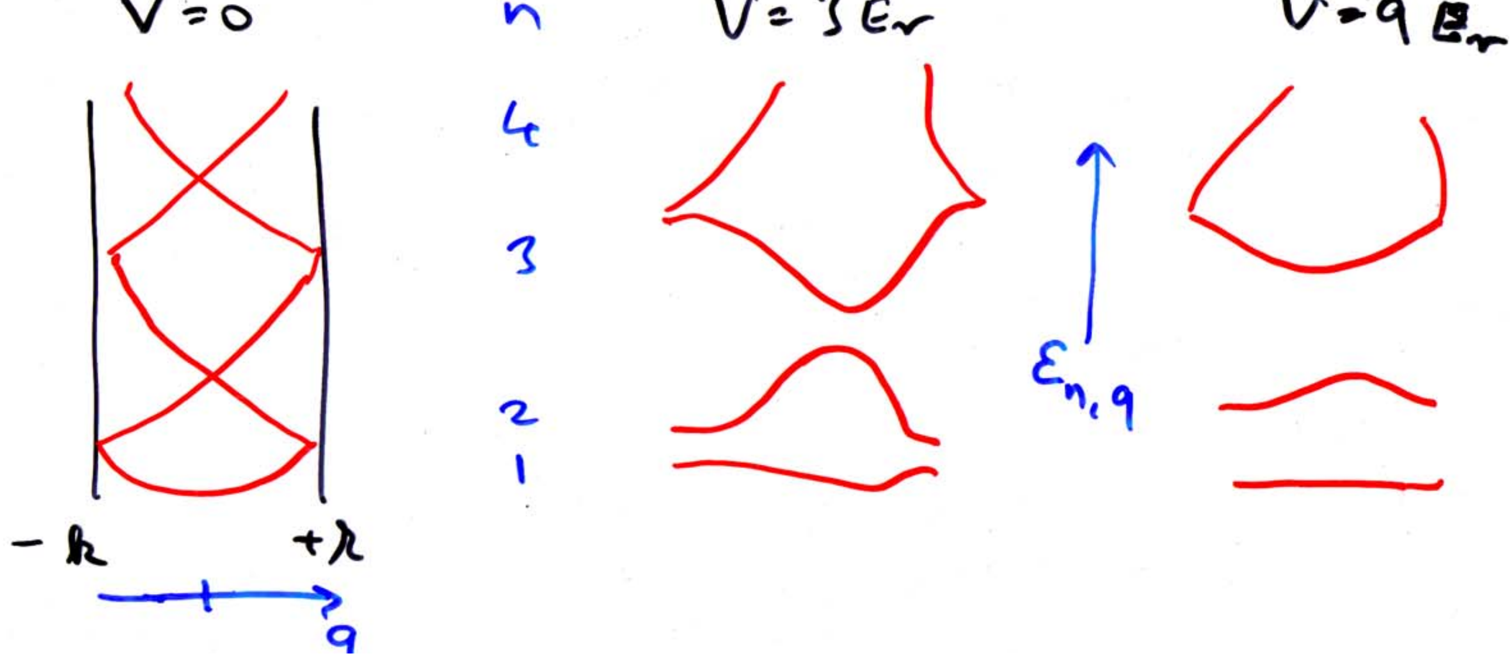
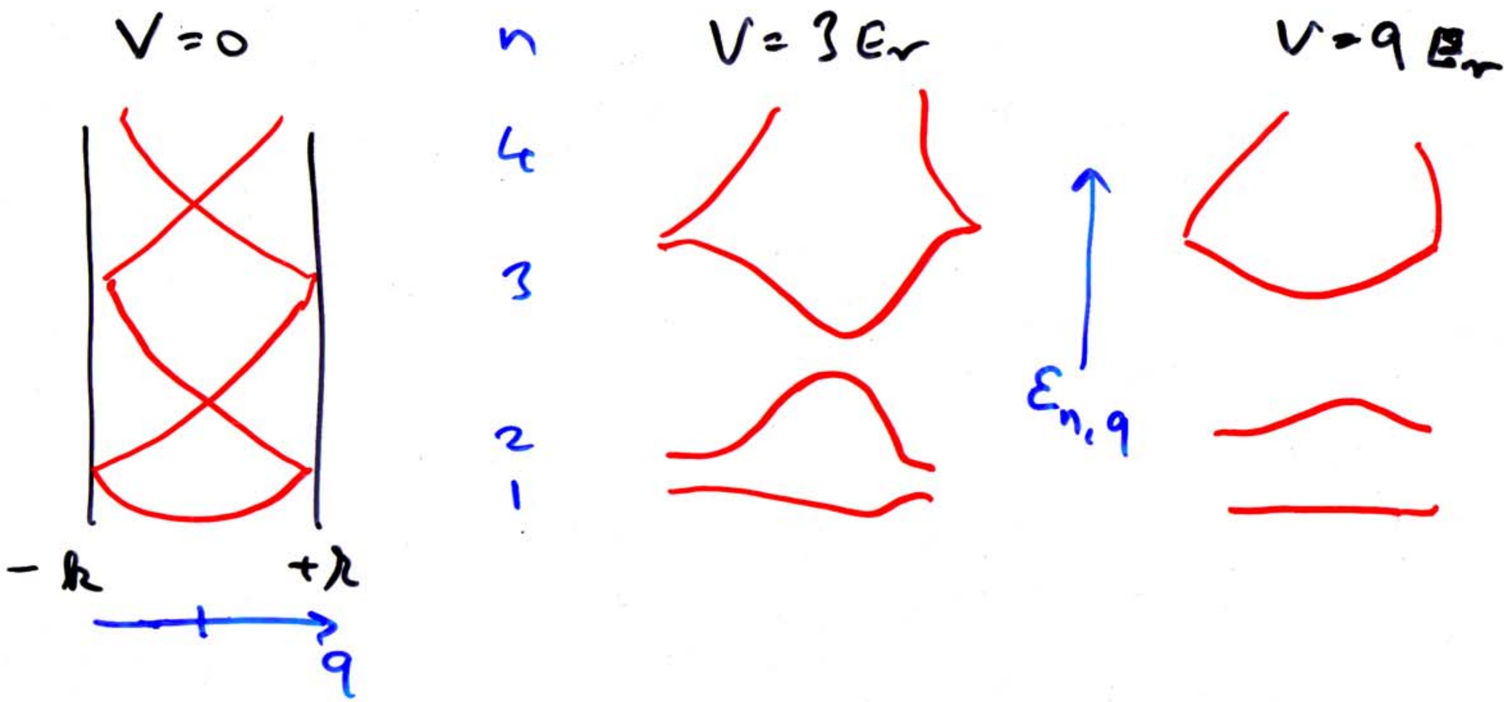
Insert into $H\psi_{q,n} = \varepsilon_{q,n}\psi_{q,n}$

$$\sum_{\ell'} e^{i(q+2\ell'\hbar k)x} \underbrace{\left[\left(\frac{(q+2\ell'\hbar k)^2}{2m} - \varepsilon_{q,n}\right)c_{\ell'}^{q,n} + \sum V_r\, c_{\ell'-r}^{q,n}\right]}_{=0} = 0$$

Set of linear equations


V = 0
n
V = 3 Er
V = 9 Er
4
3
2
1
- k
+ k
q
En, q



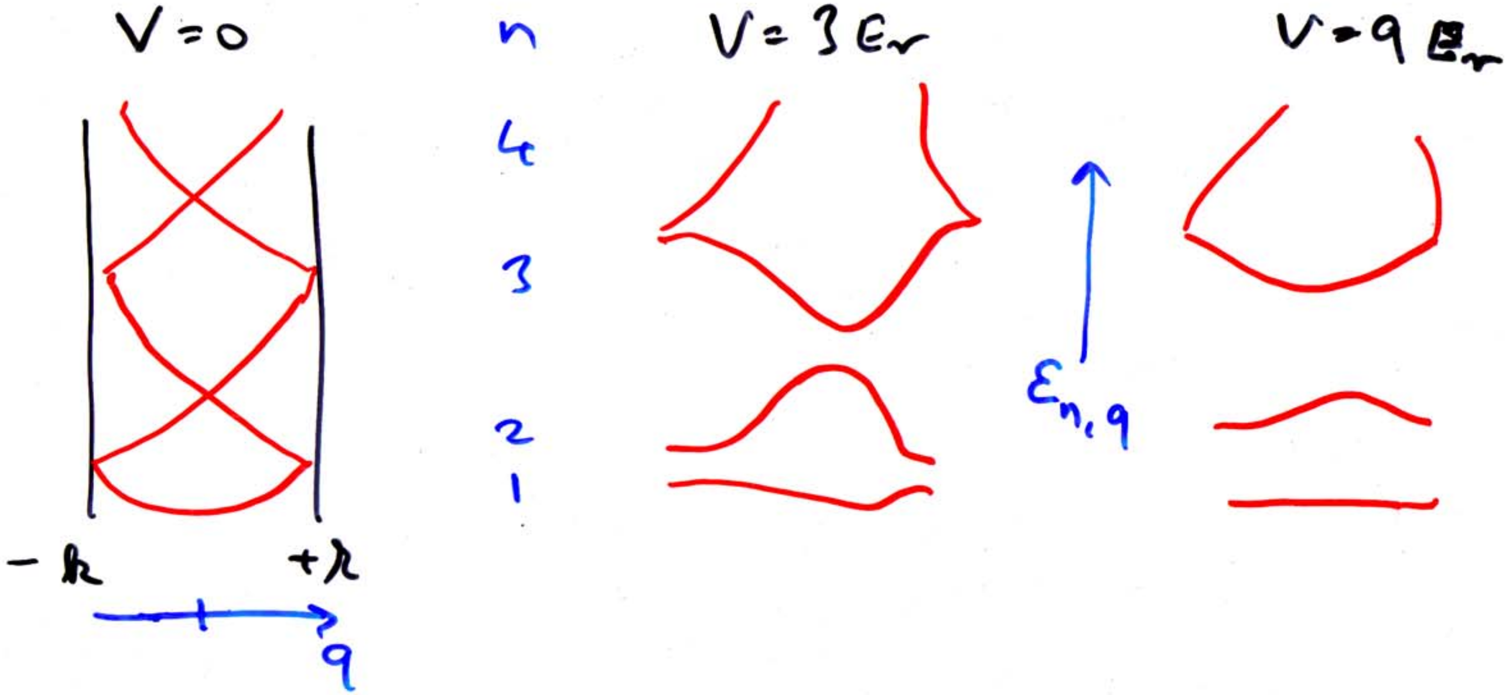
$\frac{V_0}{E_r} \gg 1$

tight binding case

$\Rightarrow$ harm. confinement at each lattice site

$E_r = \frac{\hbar^2 k^2}{2m}$

$\hbar \omega_0 = 2 E_r (V_0/E_r)^{1/2}$



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$\Rightarrow$ harm. confinement at each lattice site

$E_r = \frac{\hbar^2 k^2}{2m}$

$\hbar \omega_0 = 2 E_r (V_0/E_r)^{1/2}$

lowest band

$$\mathcal{E}_1(q) = \frac{3}{2}\hbar\omega_0 - 2J\left(\cos q_x a + \cos q_y a + \cos q_z a\right)$$

$$a = \frac{\lambda}{2} = \frac{\pi}{k}$$

J : 4 × band width

Wannier Functions (orthogonal basis set)

localized

$$w_n(x - \underbrace{x_j}_{\text{site}}) = \underbrace{N^{-1/2}}_{\text{normalization}} \sum e^{iqx_j} \psi_{q,n}(x)$$

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$$J = \int w_1(x - x_j)\left[\frac{\hbar^2}{2m}\nabla^2 + V(x)\right] w_1(x - x_l)$$

tunneling from site j to l

J "tunneling energy"

$J/\hbar$ "tunneling rate"

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tunneling from site j to ℓ

J "tunneling energy"

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deep lattice $J = \frac{4}{\sqrt{\pi}} E_r \left(\frac{V_0}{E_r}\right)^{3/4} \exp\left[-2\sqrt{V_0/E_r}\right]$

Tight binding: w_1 is Gaussian (solution for HO with freq. ω_0)

Now: Interactions

Mean Field interactions $\quad U(x) = \underbrace{\frac{4\pi\hbar^2 a_s}{m}}_{g}\,\delta(x)$

On-site interactions:

$$U = g\int [w_i(x)]^4\, d^3x = \sqrt{\frac{8}{\pi}}\, k a_s E_r \left[\frac{V_0}{E_r}\right]^{3/4}$$

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$$H_{Full} = \int d^3x\, \psi^+(x)\left(\frac{p^2}{2m} + V(x)\right)\psi(x)$$
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Wannier Functions $\hat{\psi}(x) = \sum \hat{b}_i w_1(x - x_i)$

Assume: only lowest band occupied

$$\Rightarrow H_{Full} = -\sum_{i,j} J_{ij} \, b_i^+ b_j + \frac{1}{2} \sum_{i,j,k,l} U_{ijkl} \, b_i^+ b_j^+ b_k b_l$$

$$J_{ij} = -\int dx\, w_1(x-x_i)\left(\frac{p^2}{2m} + V(x)\right) w_1(x-x_j)$$

$$U_{ijkl} = g\int dx\, w(x-x_i)\, w(x-x_j)\, w(x-x_k)\, w(x-x_l)$$

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Hubbard model

$J = J_{ij} \neq 0$ nearest neighbors

$U = U_{ijkl} \neq 0$ For $i=j=k=l$ on site

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$$\hat{H} = -J \sum_{\substack{l\,l' \\ \text{nearest} \\ \text{neighbor}}} b_l^+ b_{l'} + \frac{U}{2}\sum_l \underbrace{n_l}_{b_l^+ b_l}(n_l - 1) - \mu \sum_l n_l$$

n_l : # of bosons / site

Refs: Zwerger, J. Opt. B: Quantum Semiclass. Opt. 5, S9 (2003)

Jaksch, Zoller, Annals of Phys. 315, 52 (2005)

Two limiting cases: integer filling $\bar{n}$

$U \gg J$

ground state $|\psi_{MI}\rangle (J=0, \bar{n}) = \prod_{\ell} \left(|\bar{n}\rangle_{\ell} \right)$

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$|\Psi_{SF,N}\rangle (U=0) = \left(\frac{1}{\sqrt{M}} \sum_{\ell=1}^{M} b_{\ell}^{+} \right)^{N} |0\rangle$ M sites

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Interactions are treated only approximately
Valid only for small depletion $N - N_0$

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Goal: Find effective onsite Hamiltonian by mean-field decoupling

van Osten, van der Straten, Stoof, PR A 63 053601 (2001).

$$\hat{A}\,\hat{B} = \left(\langle A\rangle + \Delta\hat{A}\right)\left(\langle B\rangle + \Delta\hat{B}\right) \approx \langle A\rangle \Delta\hat{B} + \Delta\hat{A}\langle B\rangle + \langle A\rangle\langle B\rangle$$
$$= \langle A\rangle \hat{B} + \hat{A}\langle B\rangle - \langle A\rangle\langle B\rangle$$

Coupling between sites: tunneling $J\, b_\ell^+ b_{\ell'}$

$$b_\ell^+ b_{\ell'} \approx \langle b_\ell^+ \rangle b_{\ell'} + b_\ell^+ \langle b_{\ell'} \rangle - \langle b_\ell^+ \rangle \langle b_{\ell'} \rangle$$

Coupling between sites: tunneling $J\, b_\ell^+ b_{\ell'}$

$$b_\ell^+ b_{\ell'} \approx \underbrace{\langle b_\ell^+ \rangle}_{\psi} b_{\ell'} + b_\ell^+ \underbrace{\langle b_{\ell'} \rangle}_{\psi} - \underbrace{\langle b_\ell^+ \rangle \langle b_{\ell'} \rangle}_{\psi^2}$$

SF order parameter $\psi = \sqrt{n_\ell} = \langle b_\ell^+ \rangle = \langle b_\ell \rangle$

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z # of nearest neighbors

$$H_{eff} = -zJ\psi \sum_\ell \left(b_\ell^+ + b_\ell \right) + zJ \underbrace{M}_{\#\text{ of sites}} \psi^2 + \frac{U}{2} \sum n_\ell (n_\ell - 1) - \mu \sum n_\ell$$

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$$H_{eff} = zJ \sum H_{eff,\ell} \qquad \bar{U} = U/zJ \qquad \bar{\mu} = \mu/zJ$$

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$$H_{eff,\ell} = \frac{1}{2} \bar{U} n_\ell (n_\ell - 1) - \bar{\mu} n_\ell - \psi (b_\ell^+ + b_\ell) + \psi^2$$

$$= H^{(0)} + \psi V \qquad \text{with } V = -(b_\ell^+ + b_\ell)$$

$$H^{(0)} = \frac{1}{2} \bar{U} \hat{n} (\hat{n} - 1) - \bar{\mu} \hat{n} + \psi^2 \quad \text{diagonal in } \hat{n}$$

ground state for $H^{(0)}$

If $\bar{u}(j-1) < \bar{\mu} < \bar{u}\, j$

$\Rightarrow E_j^{(0)} = \frac{1}{2}\bar{u}\, j(j-1) - \bar{\mu}\, j$ $\quad$ j occupation #

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V: couples $\Delta n = \pm 1$

second order perturbation theory

$$E_j^{(2)} = t^2 \sum_{n \neq j} \frac{|\langle j | V | n \rangle|^2}{E_j^{(0)} - E_n^{(0)}}$$

$$= \frac{j}{\bar{u}(j-1) - \bar{\mu}} + \frac{j+1}{\bar{\mu} - \bar{u}\, j}$$

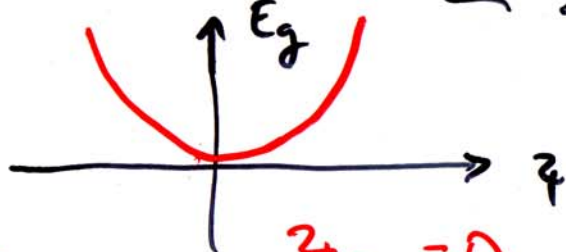
Phase transition

Landau Formalism:

$$E_g(\xi) = a_0 + a_2 \xi^2 + \mathcal{O}(\xi^4)$$

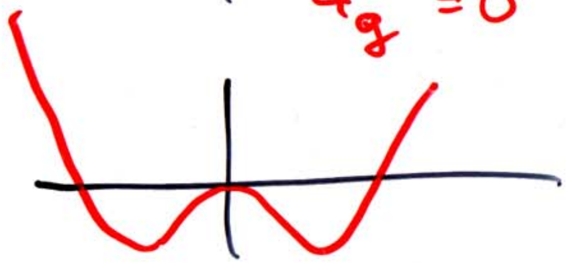
↳ >0, see 4th order perturbation theory

$a_2 > 0$



$\xi_g = 0$

$a_2 < 0$



$\xi_g \neq 0$

Phase transition for $\boxed{a_2 = 0}$

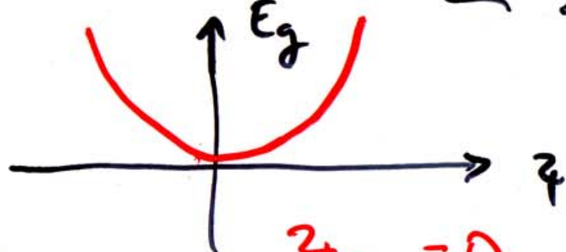
Phase transition

Landau Formalism:

$$E_g(\psi) = a_0 + a_2\psi^2 + \mathcal{O}(\psi^4)$$

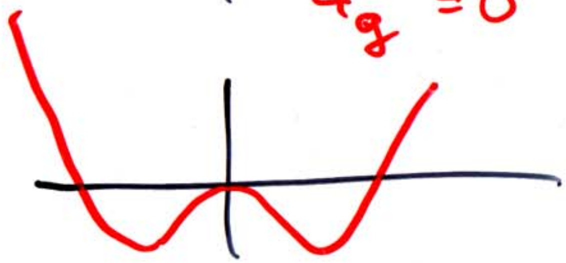
↳ > 0, see 4th order perturbation theory

$a_2 > 0$



$\psi_g = 0$

$a_2 < 0$

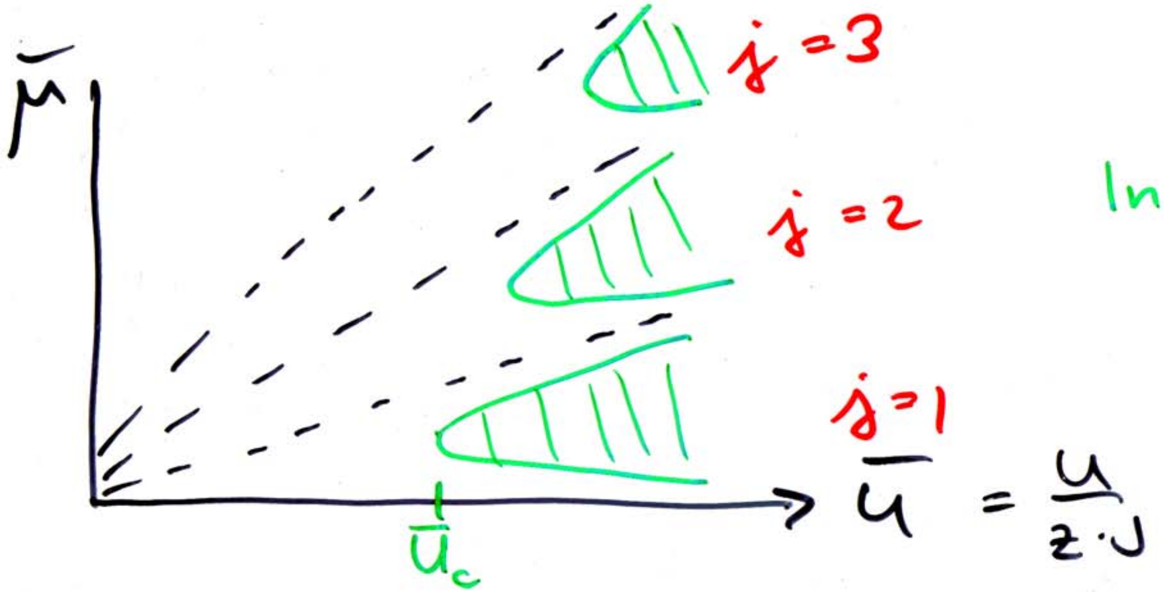


$\psi_g \neq 0$

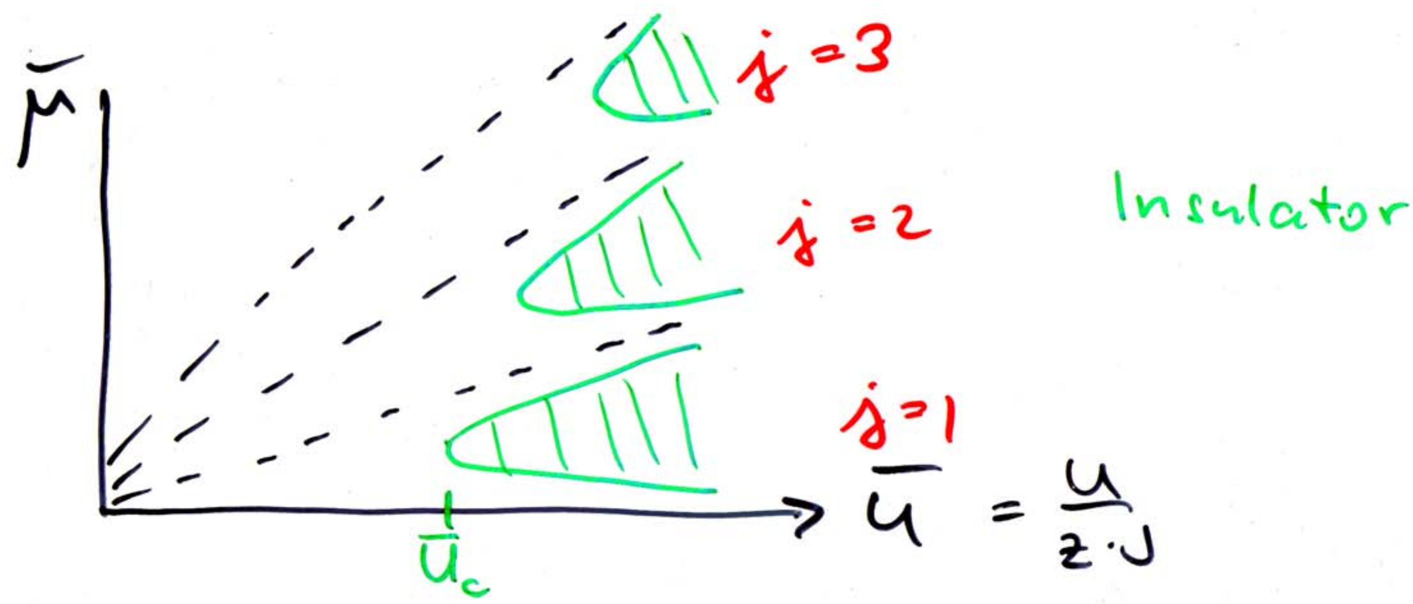
Phase transition for $\boxed{a_2 = 0}$

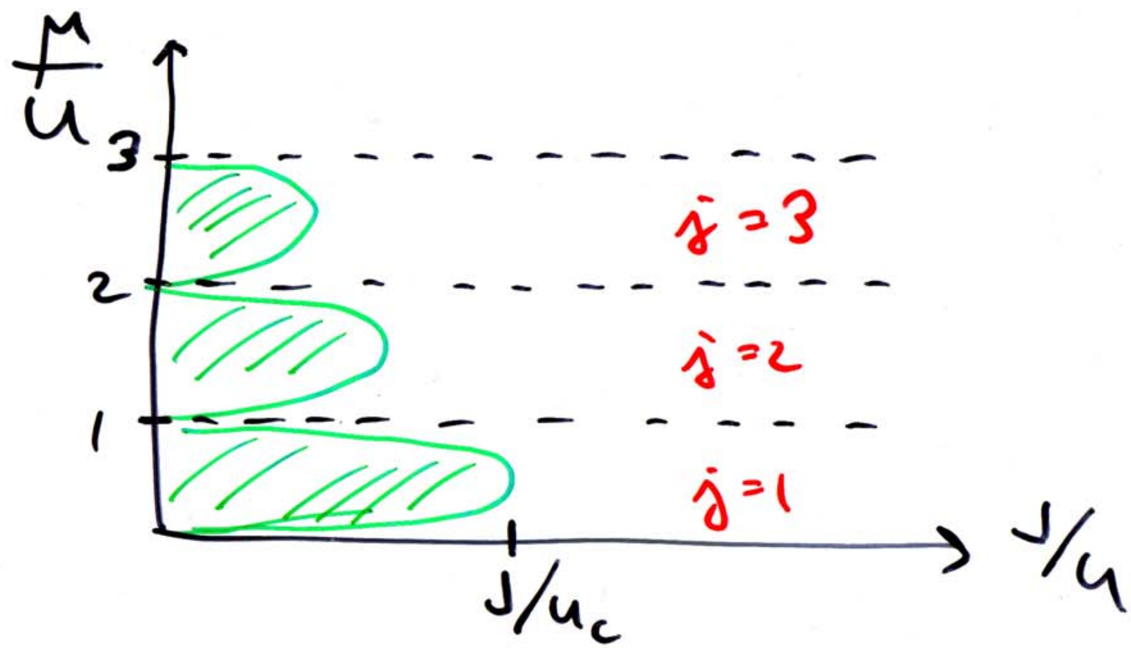
$$a_2 = \frac{j}{\bar{u}(j-1) - \bar{\mu}} + \frac{j+1}{\bar{\mu} - \bar{u}j} + 1 = 0$$

$$\bar{\mu}_\pm = \frac{1}{2}\left[\bar{u}(2j+1) - 1\right] \pm \frac{1}{2}\sqrt{\bar{u}^2 - 2\bar{u}(2j+1) + 1}$$

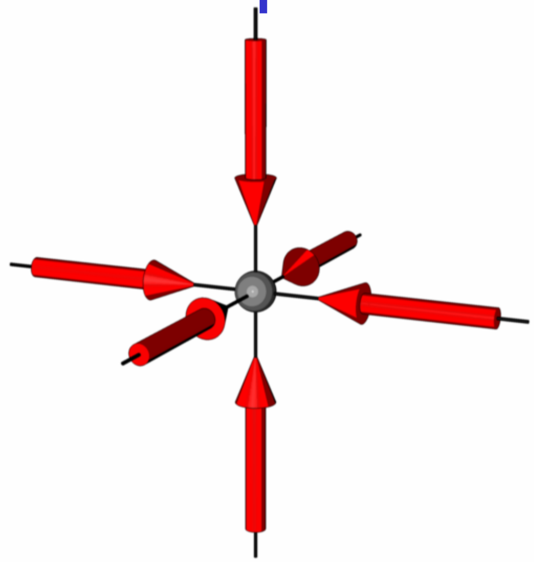


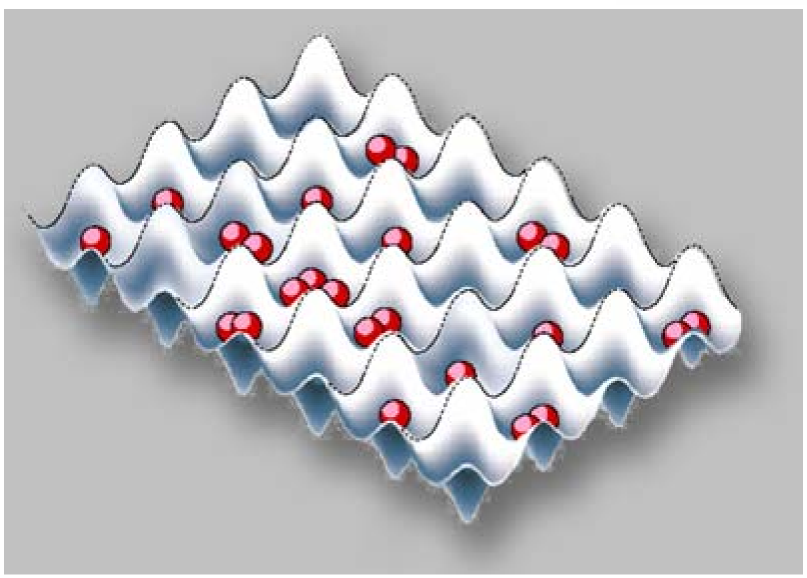
$\bar{\mu}$
$j=3$
$j=2$
$j=1$
Insulator
$\bar{U}_c$
$\bar{U} = \frac{U}{z \cdot J}$

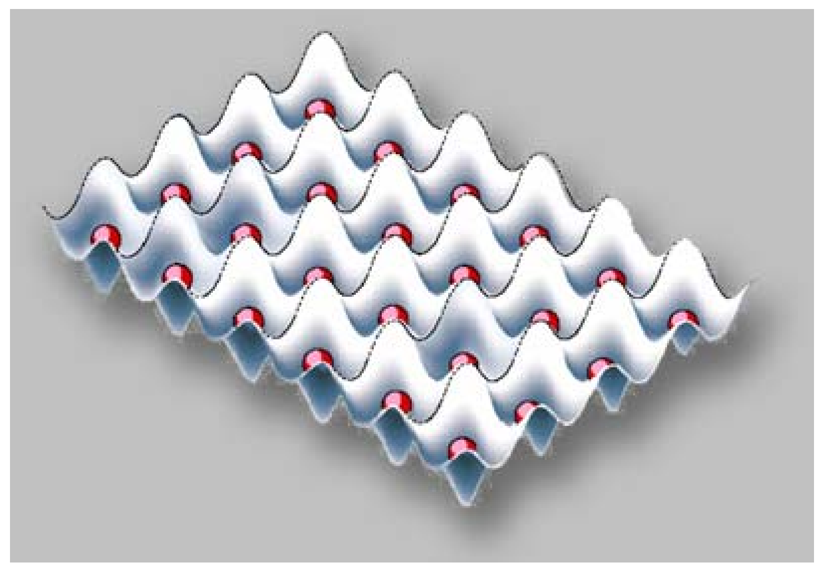

$\bar{\mu}$
$j=3$
$j=2$
$j=1$
Insulator
$\bar{U} = \frac{U}{z \cdot J}$
$\bar{U}_c$


$\frac{\mu}{U}$
3
2
1
$j=3$
$j=2$
$j=1$
J/U
J/U_c

BEC in 3D optical lattice



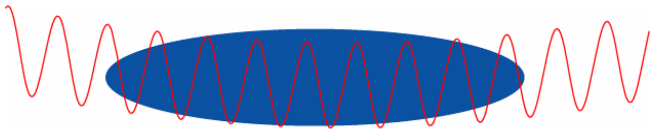




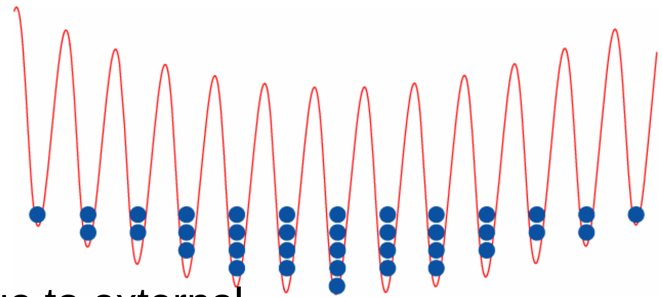
Courtesy Markus Greiner

The Superfluid-Mott Insulator transition

Shallow Lattices - Superfluid



Deep Lattices – Mott Insulator



Energy offset due to external harmonic confinement. Not in condensed matter systems.

$$\hat{H} = -J\sum_{\langle i,j\rangle} \hat{a}_i^\dagger a_j + \sum_i \frac{1}{2}U\hat{n}_i(\hat{n}_i - 1) + \sum_i (\epsilon_i - \mu)\hat{n}_i$$

tunneling term between neighboring sites

$$J = \frac{4}{\sqrt{\pi}} E_r \left(\frac{V_0}{E_r}\right)^{3/4} \exp -2\left(\frac{V_0}{E_r}\right)^{1/2}$$

on-site interaction

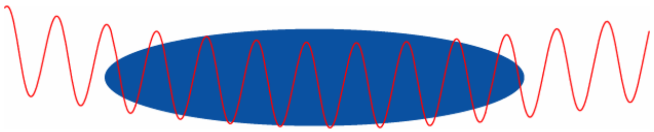
$$U = \left(\frac{4\pi\hbar^2 a}{m}\right)\int |w(x)|^4 d^3x$$

a = s-wave scattering length

Other exp: Mainz, Zurich, NIST Gaithersburg, Innsbruck, MPQ and others

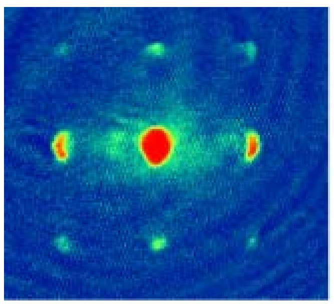
The Superfluid-Mott Insulator transition

Shallow Lattices - Superfluid

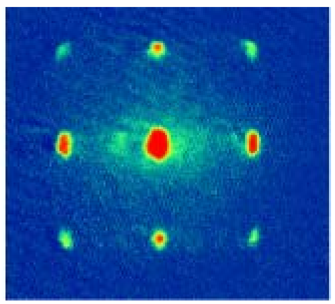


$$|\Psi_{SF}\rangle \propto \left(\sum_{i=1}^{M} \hat{a}_i^\dagger\right)^N |0\rangle$$

$$\hat{H} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger a_j + \sum_i \frac{1}{2} U \hat{n}_i(\hat{n}_i - 1) + \sum_i (\epsilon_i - \mu)\hat{n}_i$$



$5\ E_{rec}$

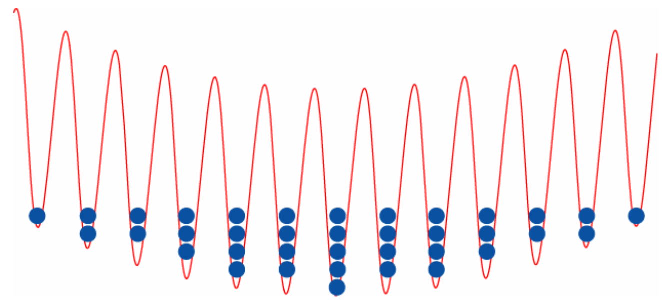


$9\ E_{rec}$

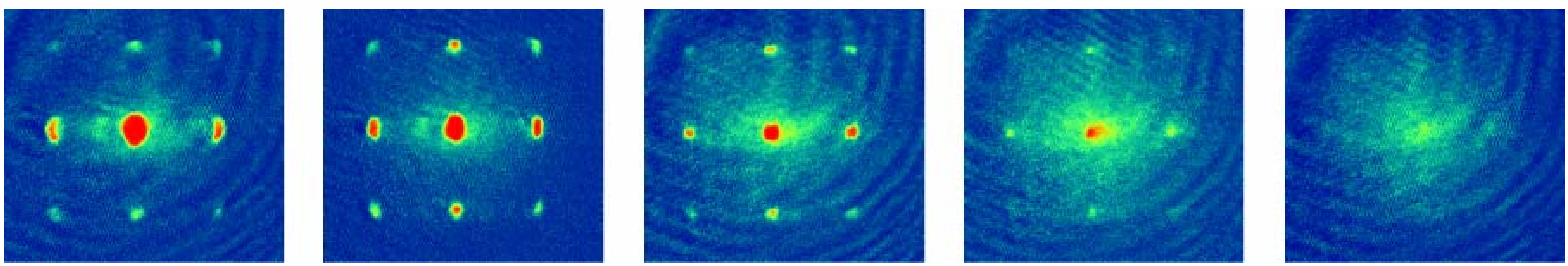
The Superfluid-Mott Insulator transition

Deep Lattices – Mott Insulator

$$|\Psi_{MI}\rangle_{J=0} \propto \prod_{i=1}^{M} (\hat{a}_i^{\dagger})^n |0\rangle$$



$$\hat{H} = -J \sum_{\langle i,j \rangle} \hat{a}_i^{\dagger} a_j + \sum_i \frac{1}{2} U \hat{n}_i (\hat{n}_i - 1) + \sum_i (\epsilon_i - \mu) \hat{n}_i$$



5 E_{rec} 9 E_{rec} 12 E_{rec} 15 E_{rec} 20 E_{rec}

As the lattice depth is increased, J decreases exponentially, and U increases. For J/U<<1, number fluctuations are suppressed, and the atoms are localized

Nanokelvin atoms are a new toolbox to address fundamental questions of many-body physics

Quantum simulations of strongly correlated, strongly interacting systems