QED

• Maxwell eq.

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{1}{\varepsilon_0} \rho(\mathbf{r}, t) \qquad (1.a)$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \qquad (1.b)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t) \qquad (1.c)$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{E}(\mathbf{r}, t) + \frac{1}{\varepsilon_0 c^2} \mathbf{j}(\mathbf{r}, t) \qquad (1.d)$$

• Spatial Fourier transform

$$i\mathbf{k} \cdot \mathscr{E}(\mathbf{k}, t) = \frac{1}{\varepsilon_0} \rho(\mathbf{k}, t) \qquad (2.a)$$

$$i\mathbf{k} \cdot \mathscr{B}(\mathbf{k}, t) = 0 \qquad (2.b)$$

$$i\mathbf{k} \times \mathscr{E}(\mathbf{k}, t) = -\frac{\partial}{\partial t} \mathscr{B}(\mathbf{k}, t) \qquad (2.c)$$

$$i\mathbf{k} \times \mathscr{B}(\mathbf{k}, t) = \frac{1}{c^2} \frac{\partial \mathscr{E}(\mathbf{k}, t)}{\partial t} + \frac{1}{\varepsilon_0 c^2} \mathbf{\jmath}(\mathbf{k}, t) \qquad (2.d)$$

• transverse and long. field decouple

$$\mathbf{E}_{\|}(\mathbf{r}, t) = \frac{-1}{4\pi \varepsilon_0} \int d^3 r' \, \rho(\mathbf{r}', t) \nabla_{\mathbf{r}} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \qquad (5.a)$$

$$\mathbf{B}_{\|}(\mathbf{r}, t) = \mathbf{0}. \qquad (5.b)$$

$$\frac{\partial}{\partial t} \mathscr{B}(\mathbf{k}, t) = -i\mathbf{k} \times \mathscr{E}_{\perp}(\mathbf{k}, t) \qquad (6.a)$$

$$\frac{\partial}{\partial t} \mathscr{E}_{\perp}(\mathbf{k}, t) = c^2 i\mathbf{k} \times \mathscr{B}(\mathbf{k}, t) - \frac{1}{\varepsilon_0} \mathbf{\jmath}_{\perp}(\mathbf{k}, t). \qquad (6.b)$$

Vector Potential

$$E(\mathbf{r}, t) = -\nabla U(\mathbf{r}, t) - \frac{\partial A(\mathbf{r}, t)}{\partial t} \qquad (7.a)$$

$$B(\mathbf{r}, t) = \nabla \times A(\mathbf{r}, t) \qquad (7.b)$$

$$\mathscr{E}(\mathbf{k}, t) = -i\mathbf{k}\mathscr{U}(\mathbf{k}, t) - \frac{\partial \mathscr{A}(\mathbf{k}, t)}{\partial t} \qquad (8.a)$$

$$\mathscr{B}(\mathbf{k}, t) = i\mathbf{k} \times \mathscr{A}(\mathbf{k}, t). \qquad (8.b)$$

Coulomb Gauge $\nabla \cdot \vec{A} = 0$

$$A_{\parallel}(\mathbf{r}, t) = 0$$

$$U(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}. \qquad (15)$$

Normal modes

$$\boldsymbol{\alpha}(\mathbf{k}, t) = \lambda(k)\left[\mathscr{E}_\perp(\mathbf{k}, t) - c\frac{\mathbf{k}}{k} \times \mathscr{B}(\mathbf{k}, t)\right]$$

$$= \lambda(k)\left[-\dot{\mathscr{A}}_\perp(\mathbf{k}, t) + i\omega\mathscr{A}_\perp(\mathbf{k}, t)\right] \qquad (16)$$

$$\dot{\boldsymbol{\alpha}}(\mathbf{k}, t) + i\omega\boldsymbol{\alpha}(\mathbf{k}, t) = \frac{i}{\sqrt{2\varepsilon_0\hbar\omega}}\boldsymbol{\jmath}_\perp(\mathbf{k}, t). \qquad (17)$$

$$\vec{A}_\perp(\mathbf{r}) = \int d^3k \sum_\varepsilon A_w\left[\vec{\varepsilon}\,\alpha_\varepsilon(k)\,e^{i\mathbf{k}\mathbf{r}} + \vec{\varepsilon}\,\alpha_\varepsilon^*(k)\,e^{-i\mathbf{k}\mathbf{r}}\right]$$

Field Quantization

$$\pi_\varepsilon(\mathbf{k}) = \varepsilon_0 \dot{A}_\varepsilon(\mathbf{k}) \quad \text{conjugate moment}$$

$$[\mathscr{A}_\varepsilon(\mathbf{k}), \pi_{\varepsilon'}^+(\mathbf{k}')] = i\hbar \delta_{\varepsilon\varepsilon'} \delta(\mathbf{k} - \mathbf{k}'). \qquad (20.a)$$

$$[a_\varepsilon(\mathbf{k}), a_{\varepsilon'}^+(\mathbf{k}')] = \delta_{\varepsilon\varepsilon'} \delta(\mathbf{k} - \mathbf{k}') \qquad (20.b)$$

• Quantum fields

$$\mathbf{A}_\perp(\mathbf{r}) = \int d^3k \sum_\varepsilon \mathscr{A}_\omega \left[\varepsilon a_\varepsilon(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} + \varepsilon a_\varepsilon^+(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \qquad (21)$$

$$\mathbf{E}_\perp(\mathbf{r}) = \int d^3k \sum_\varepsilon i\mathscr{E}_\omega \left[\varepsilon a_\varepsilon(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} - \varepsilon a_\varepsilon^+(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \qquad (22)$$

$$\mathbf{B}(\mathbf{r}) = \int d^3k \sum_\varepsilon i\mathscr{B}_\omega \left[(\boldsymbol{\kappa} \times \boldsymbol{\varepsilon}) a_\varepsilon(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} - (\boldsymbol{\kappa} \times \boldsymbol{\varepsilon}) a_\varepsilon^+(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \qquad (23)$$

○ Particle Operators

$$[\mathbf{r}_i, \mathbf{P}_j] = i\hbar \delta_{ij}$$

$$m_\alpha \mathbf{v}_\alpha = \mathbf{p}_\alpha - q_\alpha \mathbf{A}_\perp(\mathbf{r}_\alpha) \qquad (37)$$

$$\rho(\mathbf{r}) = \sum_\alpha q_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha) \qquad (38.a)$$

$$\mathbf{j}(\mathbf{r}) = \sum_\alpha q_\alpha \mathbf{v}_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha). \qquad (38.b)$$

Hamiltonian

$$H = \sum_{\alpha} \frac{1}{2m_{\alpha}} \left[\mathbf{p}_{\alpha} - q_{\alpha} \mathbf{A}_{\perp}(\mathbf{r}_{\alpha}) \right]^2 +$$
$$+ \sum_{\alpha} \left(-g_{\alpha} \frac{q_{\alpha}}{2m_{\alpha}} \right) \mathbf{S}_{\alpha} \cdot \mathbf{B}(\mathbf{r}_{\alpha}) + V_{\text{Coul}} + H_R. \qquad (40)$$

$$V_{\text{Coul}} = \frac{\varepsilon_0}{2} \int d^3r \, \mathbf{E}_{\parallel}^2(\mathbf{r}) = \frac{\varepsilon_0}{2} \int d^3k \, |\mathscr{E}_{\parallel}(\mathbf{k})|^2 \qquad (41)$$

$$V_{\text{Coul}} = \frac{1}{2\varepsilon_0} \int d^3k \, \frac{\rho^*(\mathbf{k})\rho(\mathbf{k})}{k^2} = \sum_{\alpha} \varepsilon_{\text{Coul}}^{\alpha} + \frac{1}{8\pi\varepsilon_0} \sum_{\alpha \neq \beta} \frac{q_{\alpha}q_{\beta}}{|\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}|}. \quad (42)$$

$$\varepsilon_{\text{Coul}}^{\alpha} = \frac{q_{\alpha}^2}{2\varepsilon_0} \int \frac{d^3k}{(2\pi)^3 k^2} = \frac{q_{\alpha}^2}{4\varepsilon_0\pi^2} \int_0^{k_c} dk = \frac{q_{\alpha}^2 k_c}{4\varepsilon_0\pi^2}. \qquad (43)$$

Coulomb self energy

$$H_R = \frac{\varepsilon_0}{2} \int d^3r \left[\mathbf{E}_{\perp}^2(\mathbf{r}) + c^2 \mathbf{B}^2(\mathbf{r}) \right] \qquad (44)$$

$$H_R = \sum_i \hbar\omega_i \left(a_i^+ a_i + \tfrac{1}{2} \right). \qquad (45)$$
$$(N_i + \tfrac{1}{2})$$

• Splitting the Hamiltonian

$$H = H_P + H_R + H_I \qquad (49)$$

$$H_P = \sum_\alpha \frac{\mathbf{p}_\alpha^2}{2m_\alpha} + V_{\text{Coul}} \qquad (50)$$

$$H_I = H_{I1} + H_{I2} + H_{I1}^S \qquad (51)$$

$$H_{I1} = -\sum_\alpha \frac{q_\alpha}{m_\alpha} \mathbf{p}_\alpha \cdot \mathbf{A}_\perp(\mathbf{r}_\alpha) \qquad (52)$$

$$H_{I1}^S = -\sum_\alpha g_\alpha \frac{q_\alpha}{2m_\alpha} \mathbf{S}_\alpha \cdot \mathbf{B}(\mathbf{r}_\alpha) \qquad (53)$$

$$H_{I2} = \sum_\alpha \frac{q_\alpha^2}{2m_\alpha} \mathbf{A}_\perp^2(\mathbf{r}_\alpha). \qquad (54)$$

Electric Dipole Approx.

$$\tau \ll \lambda$$

$$H = \sum_\alpha \frac{1}{2m_\alpha}\left[\mathbf{p}_\alpha - q_\alpha \mathbf{A}_\perp(\mathbf{0})\right]^2 + V_{\text{Coul}} + \sum_j \hbar\omega_j\left(a_j^+ a_j + \tfrac{1}{2}\right). \quad (71)$$

$$\vec{d} = \sum_\alpha q_\alpha \vec{r}_\alpha$$

$$T = \exp\left[-\frac{i}{\hbar}\mathbf{d}\cdot\mathbf{A}_\perp(\mathbf{0})\right] = \exp\left\{\sum_j\left(\lambda_j^* a_j - \lambda_j a_j^+\right)\right\} \quad (72)$$

$$H' = THT^+$$

$$= \sum_\alpha \frac{\mathbf{p}_\alpha^2}{2m_\alpha} + V_{\text{Coul}} + \varepsilon_{\text{dip}} + \sum_j \hbar\omega_j\left(a_j^+ a_j + \tfrac{1}{2}\right) -$$

$$- \mathbf{d}\cdot\sum_j \mathscr{E}_{\omega_j}\left[ia_j\boldsymbol{\varepsilon}_j - ia_j^+\boldsymbol{\varepsilon}_j\right] \quad (75)$$

$$\varepsilon_{\text{dip}} = \sum_j \frac{1}{2\varepsilon_0 L^3}\left(\boldsymbol{\varepsilon}_j\cdot\mathbf{d}\right)^2. \quad (76)$$

$$\mathbf{v}_\alpha' = T\mathbf{v}_\alpha T^+ = \frac{\mathbf{p}_\alpha}{m_\alpha} \quad (78)$$

$$H' = THT^+$$

$$= \sum_\alpha \frac{\mathbf{p}_\alpha^2}{2m_\alpha} + V_{\text{Coul}} + \varepsilon_{\text{dip}} + \sum_j \hbar\omega_j\left(a_j^+ a_j + \tfrac{1}{2}\right) -$$

$$- \mathbf{d} \cdot \sum_j \mathscr{E}_{\omega_j}\left[ia_j\boldsymbol{\varepsilon}_j - ia_j^+\boldsymbol{\varepsilon}_j\right] \qquad (75)$$

$$\varepsilon_{\text{dip}} = \sum_j \frac{1}{2\varepsilon_0 L^3}\left(\boldsymbol{\varepsilon}_j \cdot \mathbf{d}\right)^2. \qquad (76)$$

$$\mathbf{v}_\alpha' = T\mathbf{v}_\alpha T^+ = \frac{\mathbf{p}_\alpha}{m_\alpha} \qquad (78)$$

$$\mathbf{D}'(\mathbf{r})/\varepsilon_0 = \mathbf{E}'_\perp(\mathbf{r}) + \frac{1}{\varepsilon_0}\mathbf{P}'_\perp(\mathbf{r})$$

$$= \mathbf{E}_\perp(\mathbf{r})$$

$$= i\sum_j \mathscr{E}_{\omega_j}\left(a_j\boldsymbol{\varepsilon}_j\, e^{i\mathbf{k}_j\cdot\mathbf{r}} - a_j^+\boldsymbol{\varepsilon}_j\, e^{-i\mathbf{k}_j\cdot\mathbf{r}}\right). \qquad (89)$$

$$\mathbf{E}'_\perp(\mathbf{r}) = T\mathbf{E}_\perp(\mathbf{r})T^+$$

$$= \sum_j \mathscr{E}_{\omega_j}\left[i(a_j + \lambda_j)\boldsymbol{\varepsilon}_j\, e^{i\mathbf{k}_j\cdot\mathbf{r}} + \text{h.c.}\right]$$

$$= \mathbf{E}_\perp(\mathbf{r}) - \frac{1}{\varepsilon_0}\mathbf{P}_\perp(\mathbf{r}) \qquad (81)$$

$$\mathbf{P}(\mathbf{r}) = \mathbf{d}\,\delta(\mathbf{r}) \qquad (82)$$

$$H_I' = -\mathbf{d} \cdot \mathbf{D}'(0)/\varepsilon_0$$

$$H_I' = -\mathbf{d} \cdot \mathbf{E}_\perp(0)$$