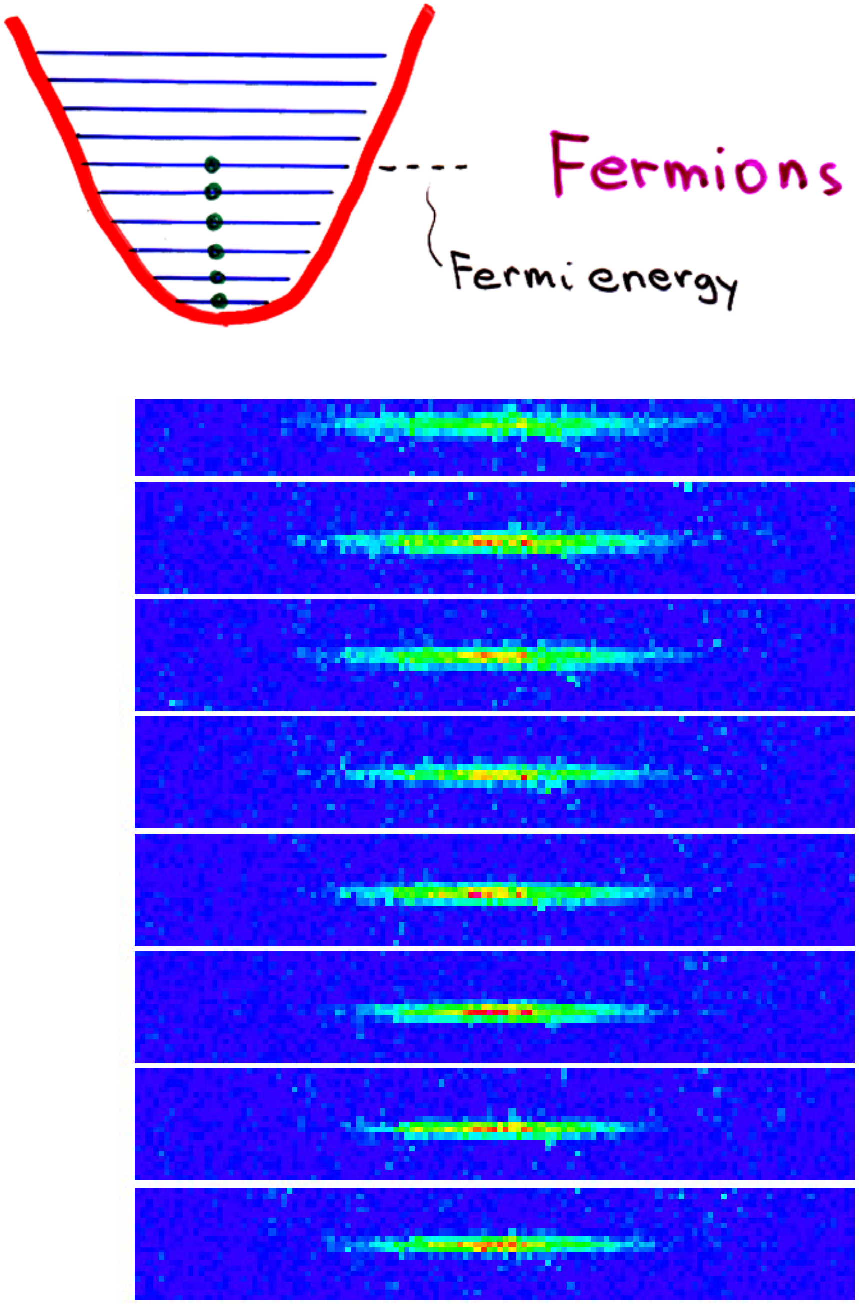
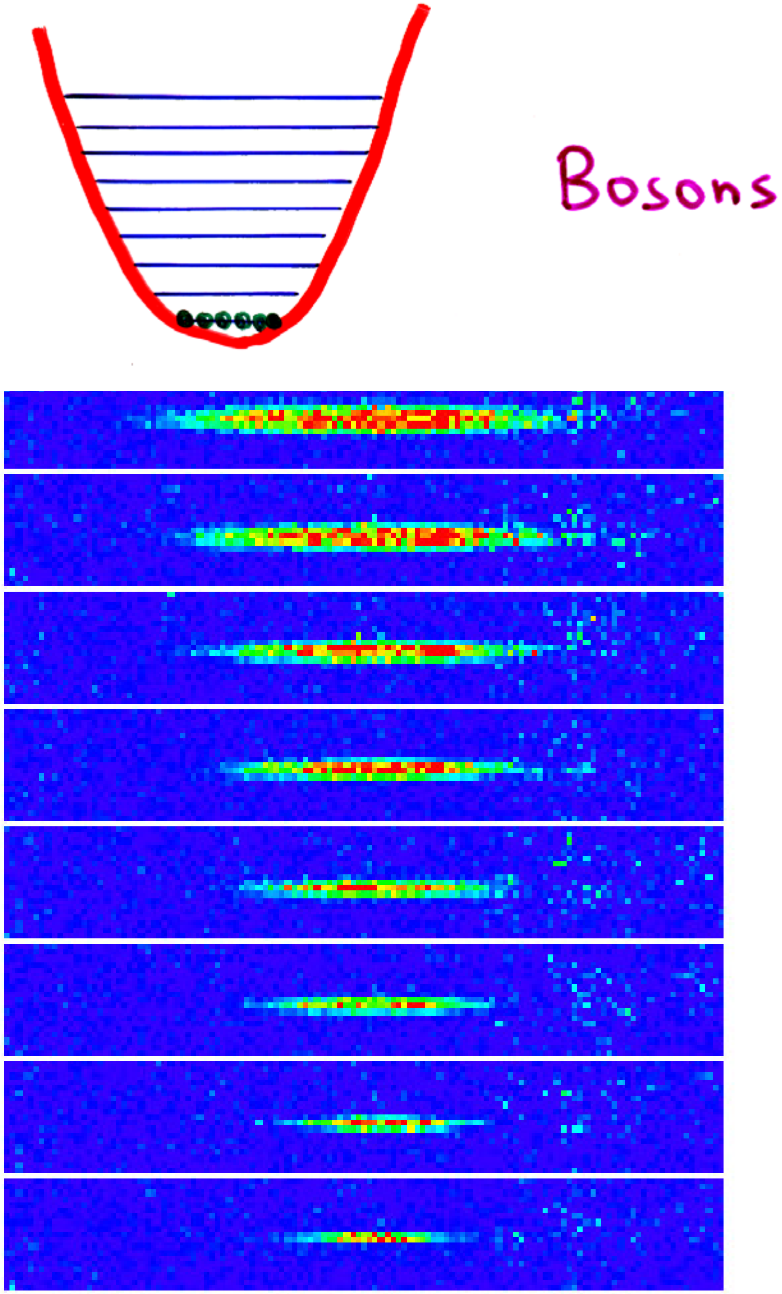
# Cold fermions

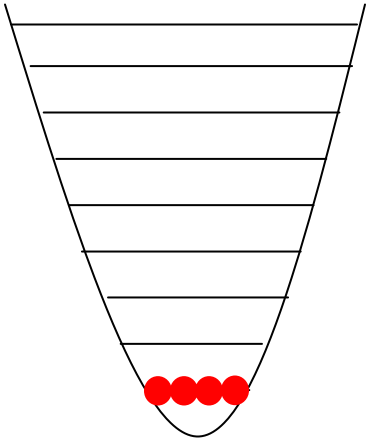
**Lithium**



**Sodium**



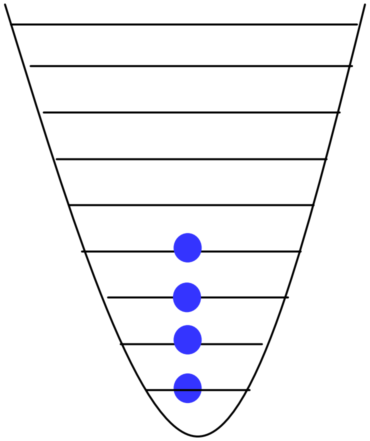
# At absolute zero temperature …



Bosons

Particles with an even number of protons, neutrons and electrons

Bose-Einstein condensation
$\Rightarrow$ atoms as waves
$\Rightarrow$ superfluidity

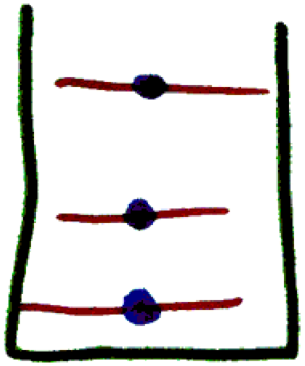


Fermions

Particles with an odd number of protons, neutrons and electrons

Fermi sea:
$\Rightarrow$ Atoms are not coherent
$\Rightarrow$ No superfluidity

# Fermions in a box

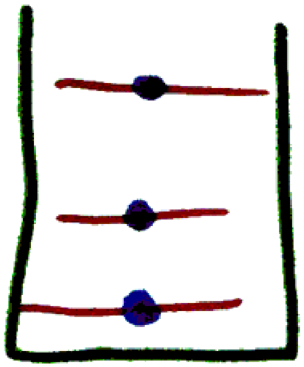


$$p_F = \hbar\left(6\pi^2 n\right)^{1/3}$$

$$E_F = p_F^2/2m$$

$$n = \left(\frac{E_F}{2m}\right)^{3/2}\frac{1}{6\pi^2\hbar^3}$$

## Fermions in a box



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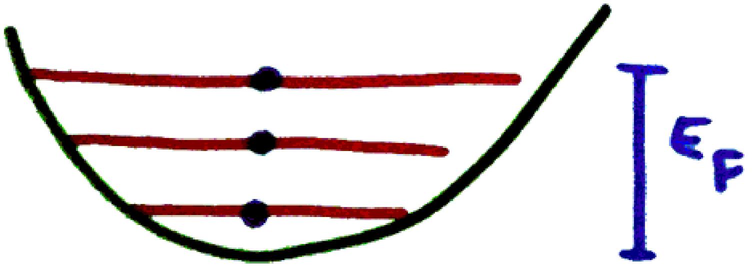
## Fermions in an HO



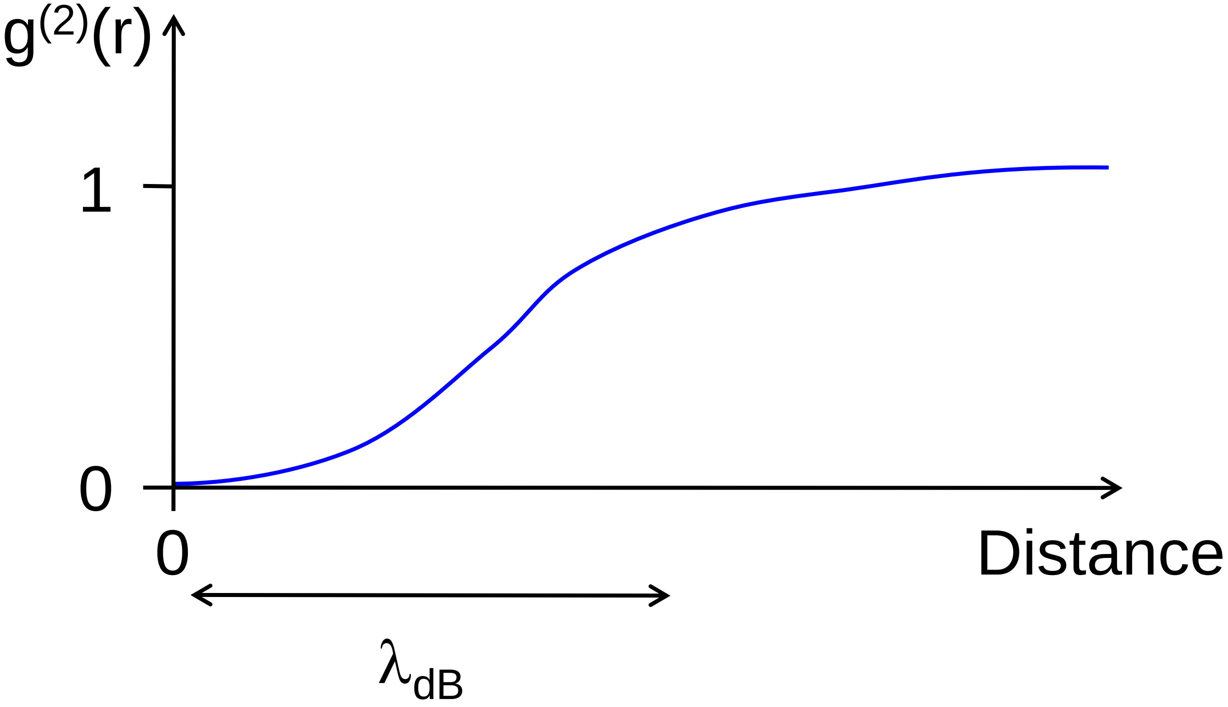
$$E_F = (6N)^{1/3}\hbar\omega$$

$$n(r) = \left(\frac{E_F - V(r)}{2m}\right)^{3/2}\frac{1}{6\pi^2\hbar^3}$$

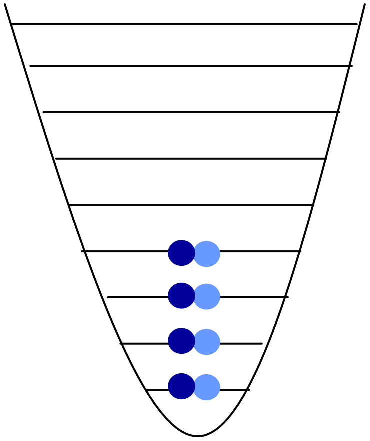
local density approximation

# Freezing out of collisions

Pair correlations in a Fermi gas:

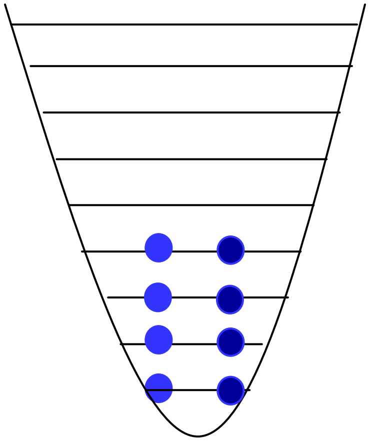


No interactions if range of potential is < $\lambda_{dB}$



Pairs of fermions

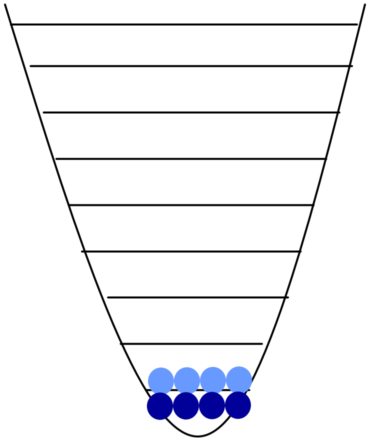
Particles with an even number of protons, neutrons and electrons



Two kinds of fermions

Fermi sea:
⇒ Atoms are not coherent
⇒ No superfluidity

# At absolute zero temperature …



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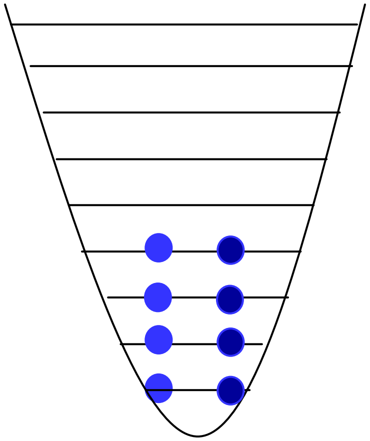
Particles with an even number of protons, neutrons and electrons

Bose-Einstein condensation

⇒ atoms as waves

⇒ superfluidity



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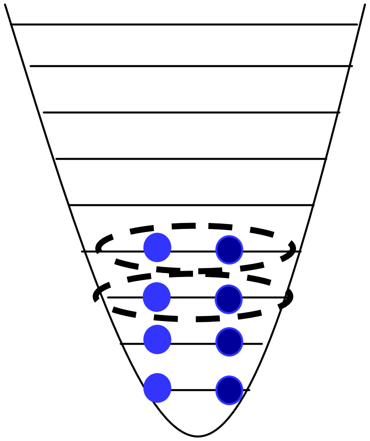
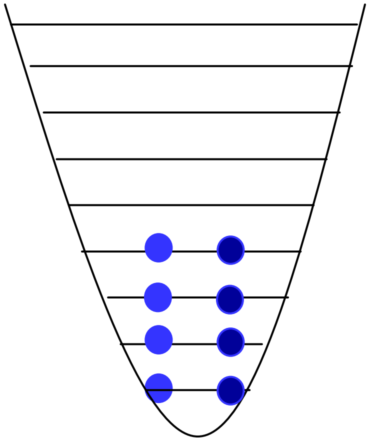


Weak attractive interactions

Cooper pairs
larger than interatomic distance
momentum correlations
⇒ BCS superfluidity



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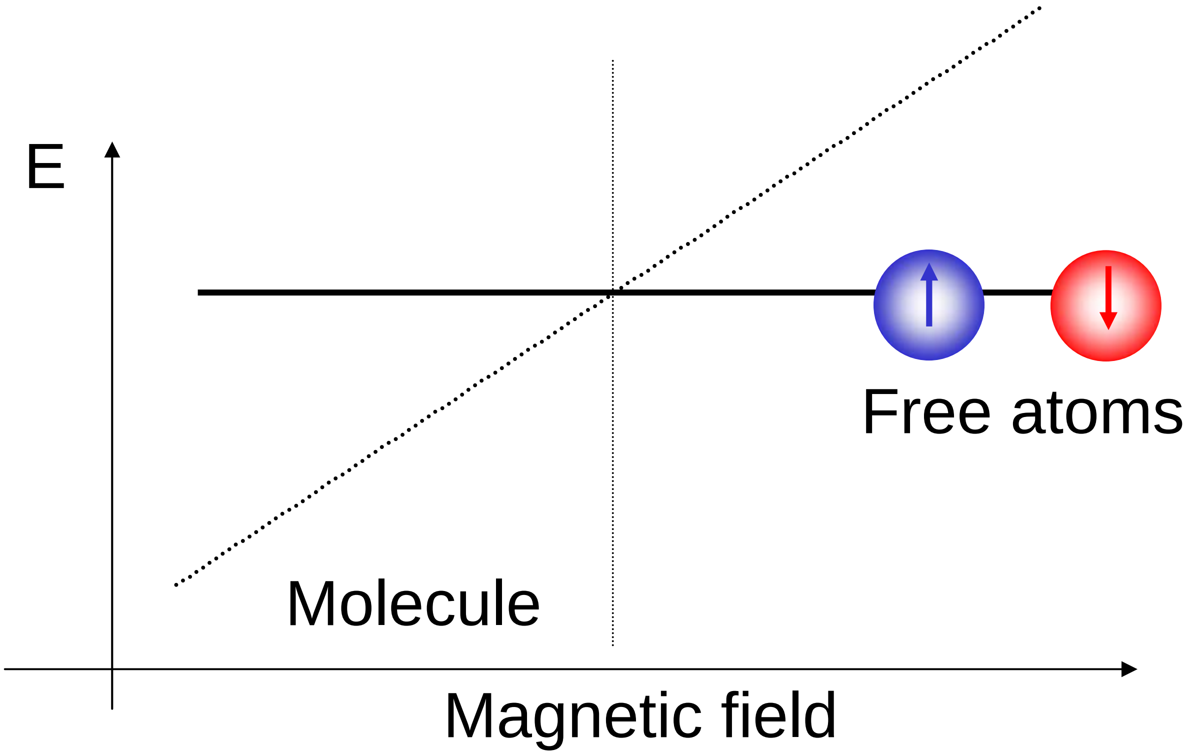
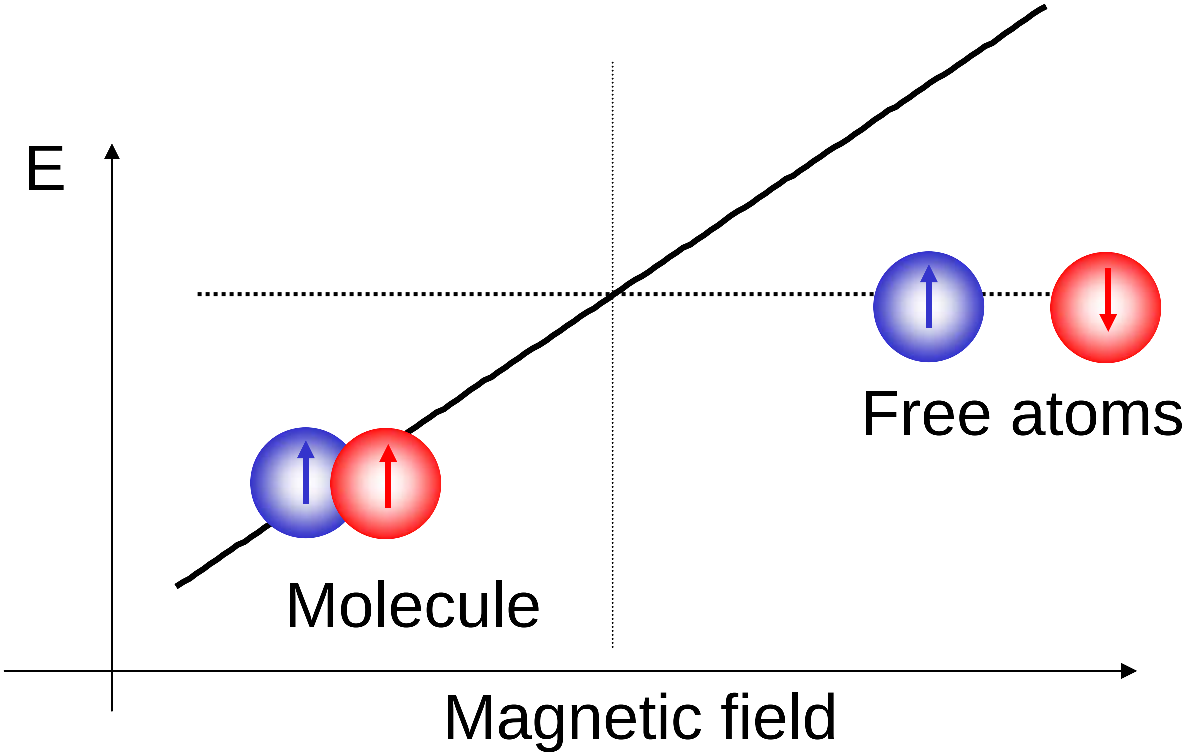


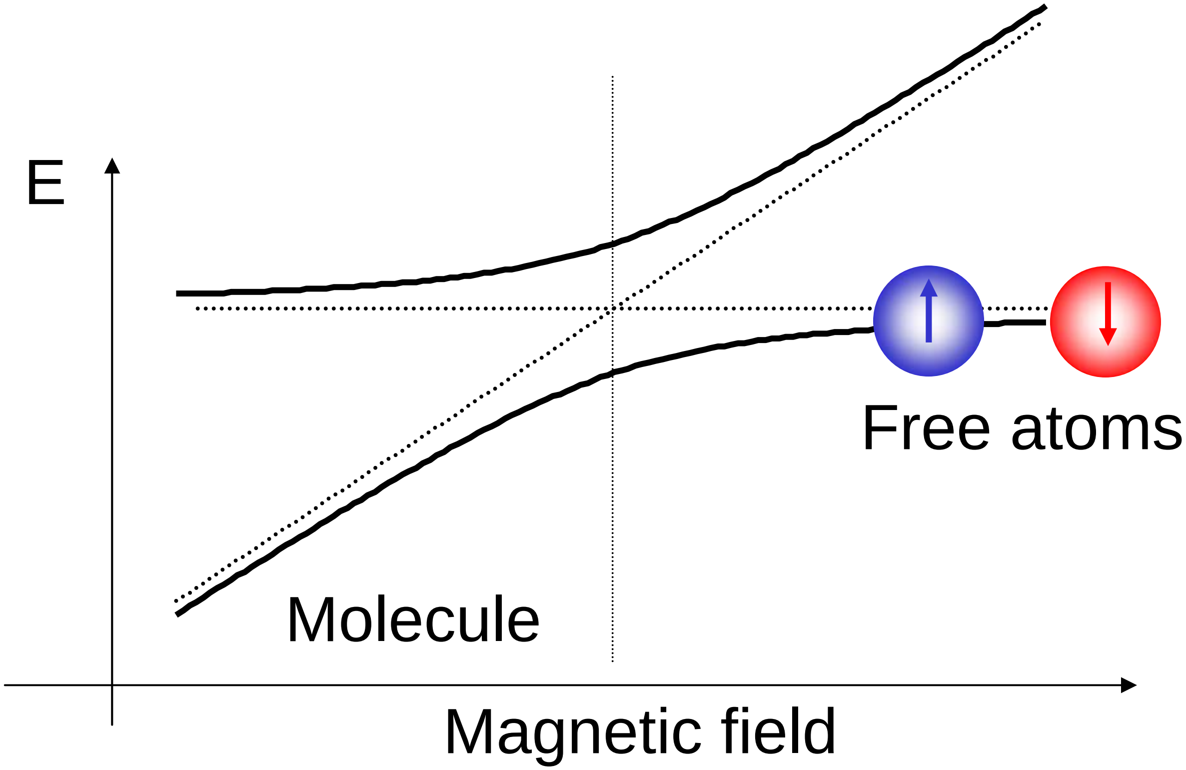
Feshbach resonance

Disclaimer: Drawing is schematic
and does not distinguish nuclear
and electron spin.



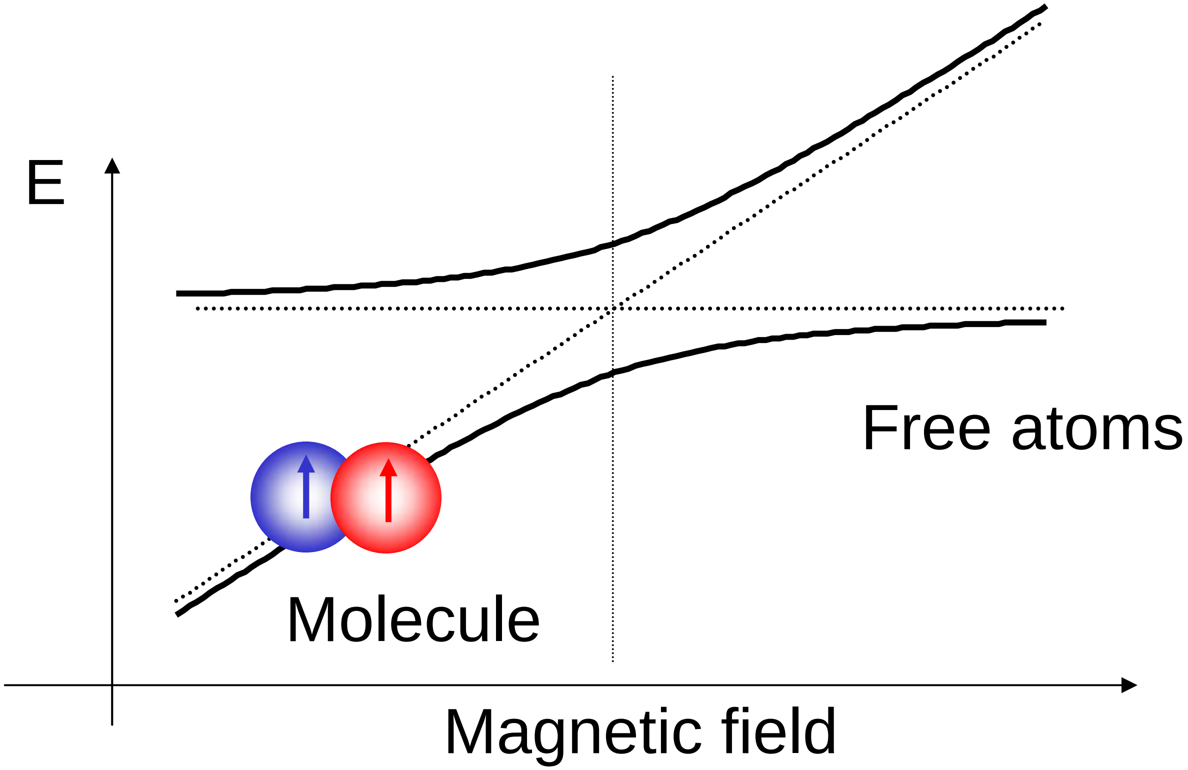
Feshbach resonance

Two atoms ….



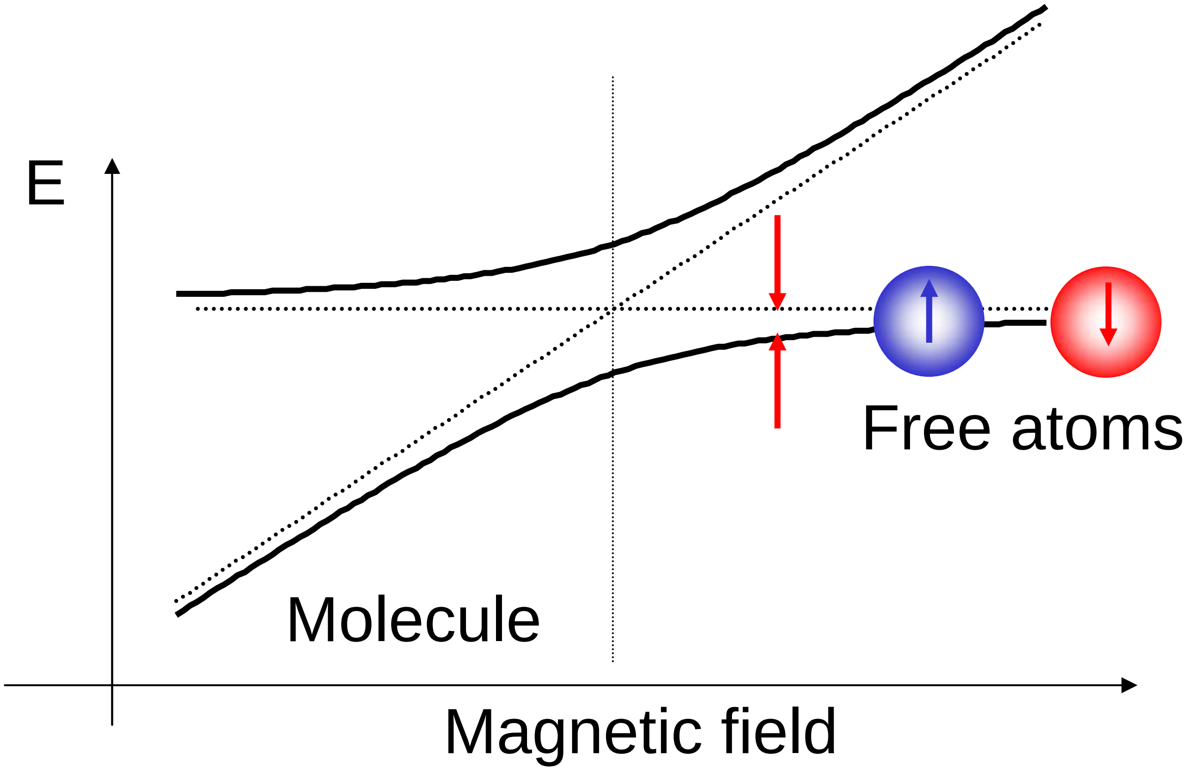
Feshbach resonance

… form a stable molecule



Feshbach resonance

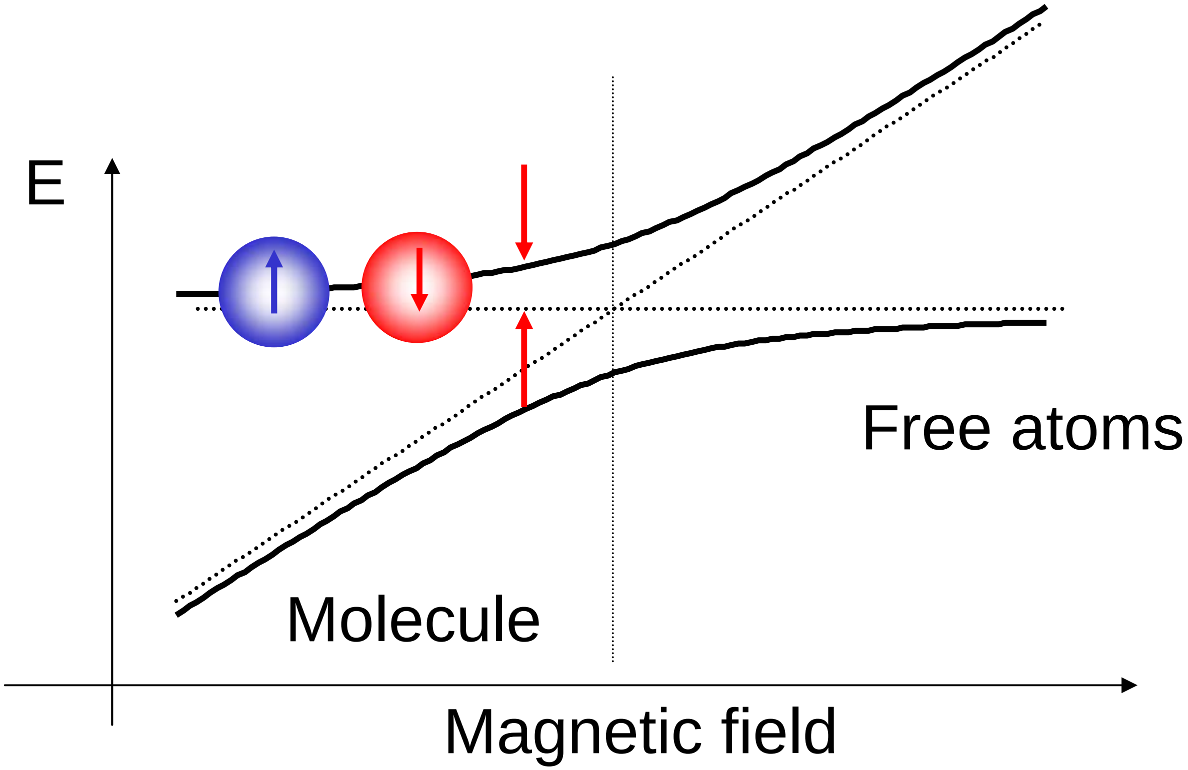
Atoms attract each other



Feshbach resonance

Atoms repel each other

Atoms attract each other



Feshbach resonance

Atoms repel each other

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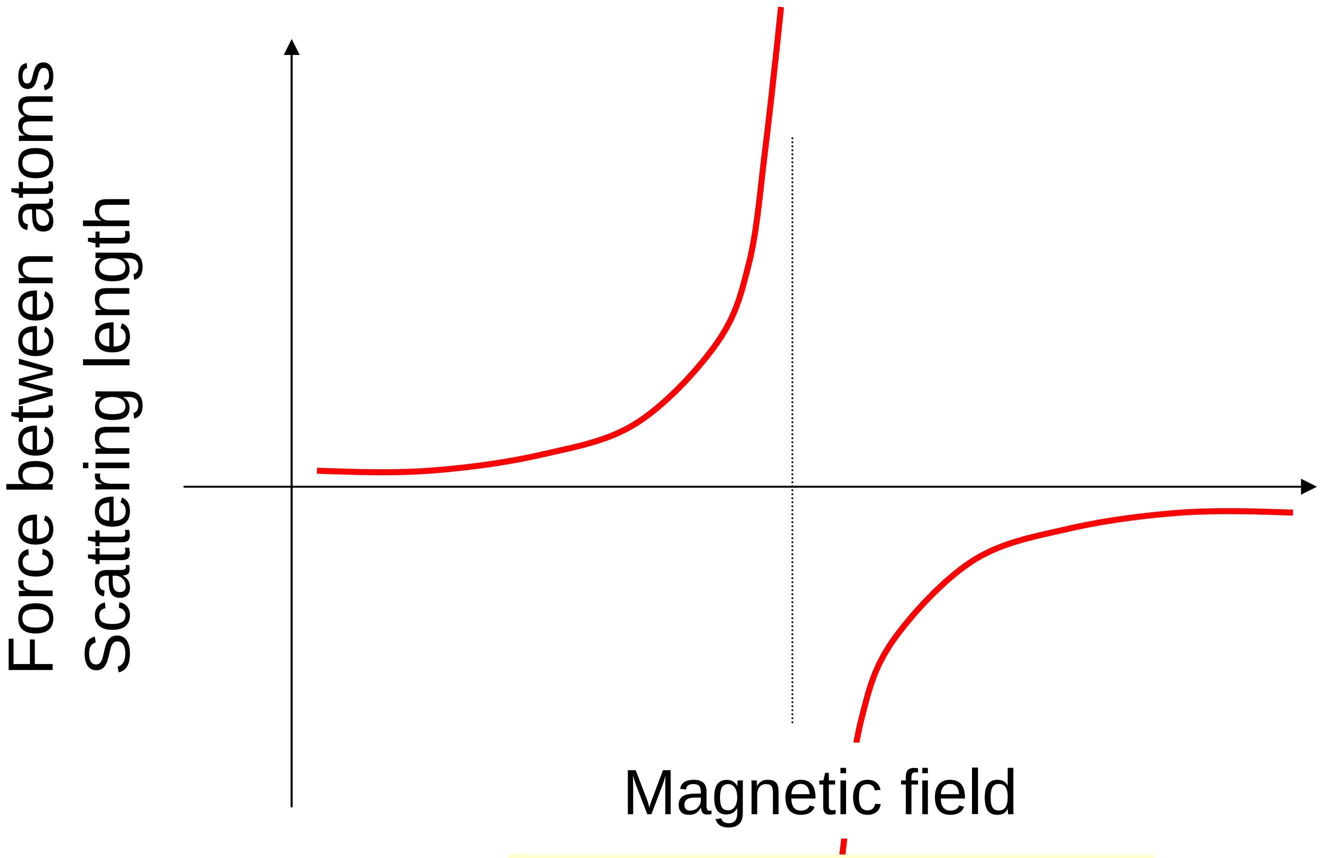
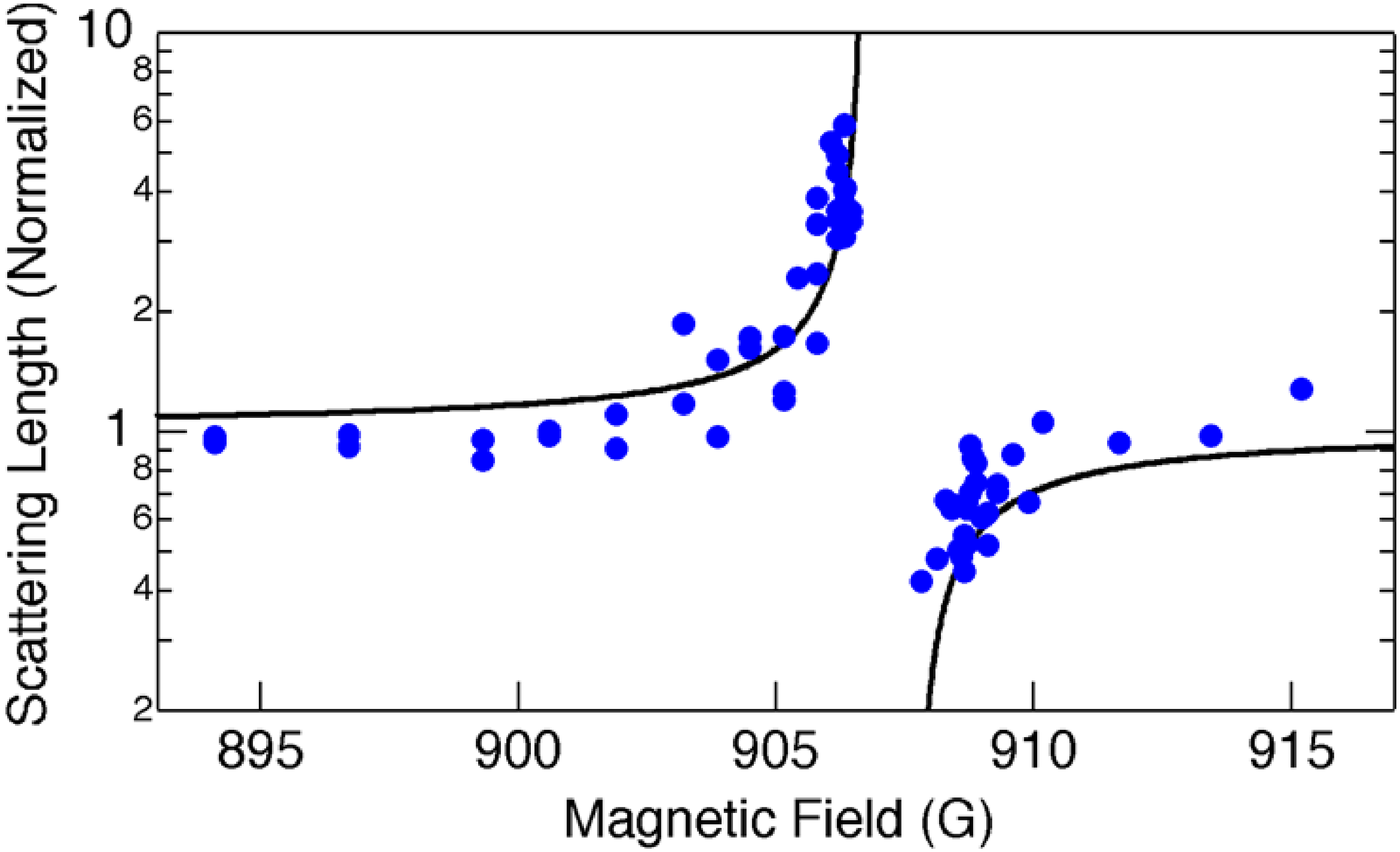
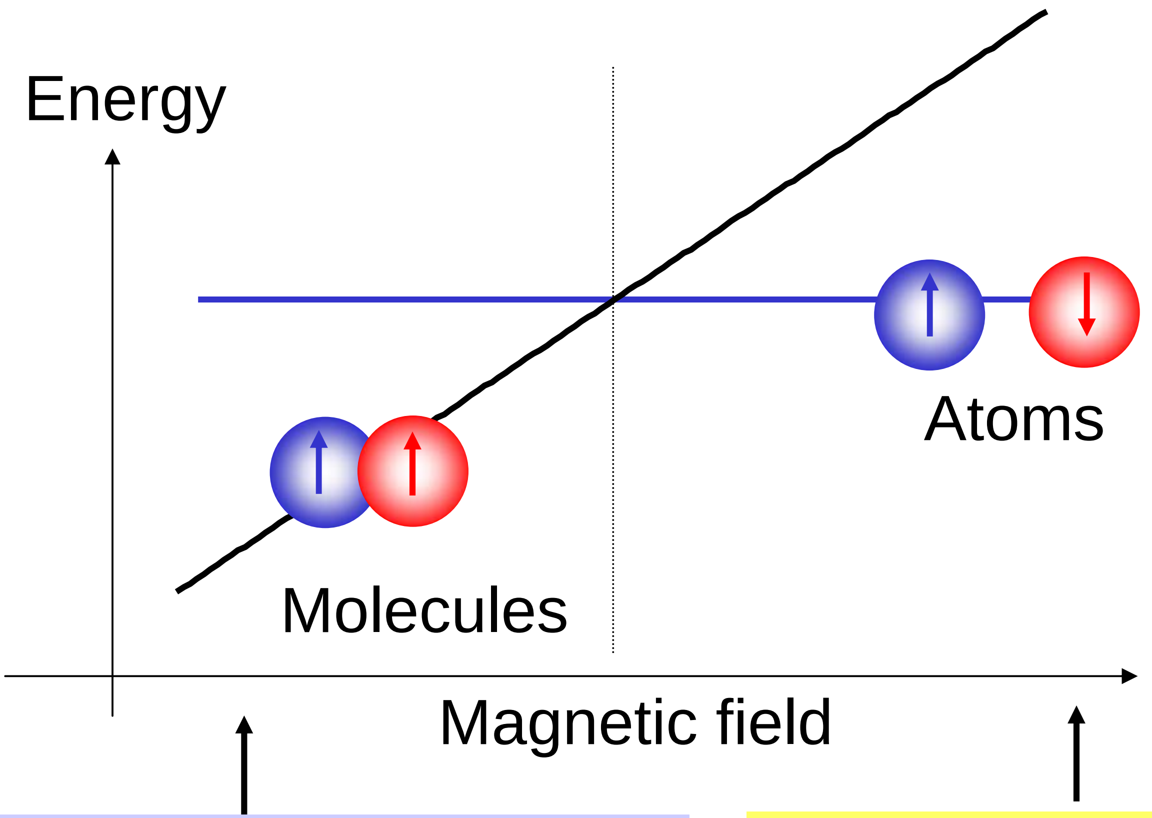


Feshbach resonance

# Observation of a Feshbach resonance



S. Inouye, M.R. Andrews, J. Stenger, H.-J. Miesner, D.M. Stamper-Kurn, WK,
Nature **392** (1998).



Atoms form stable molecules

**Atoms repel each other**
**a>0**

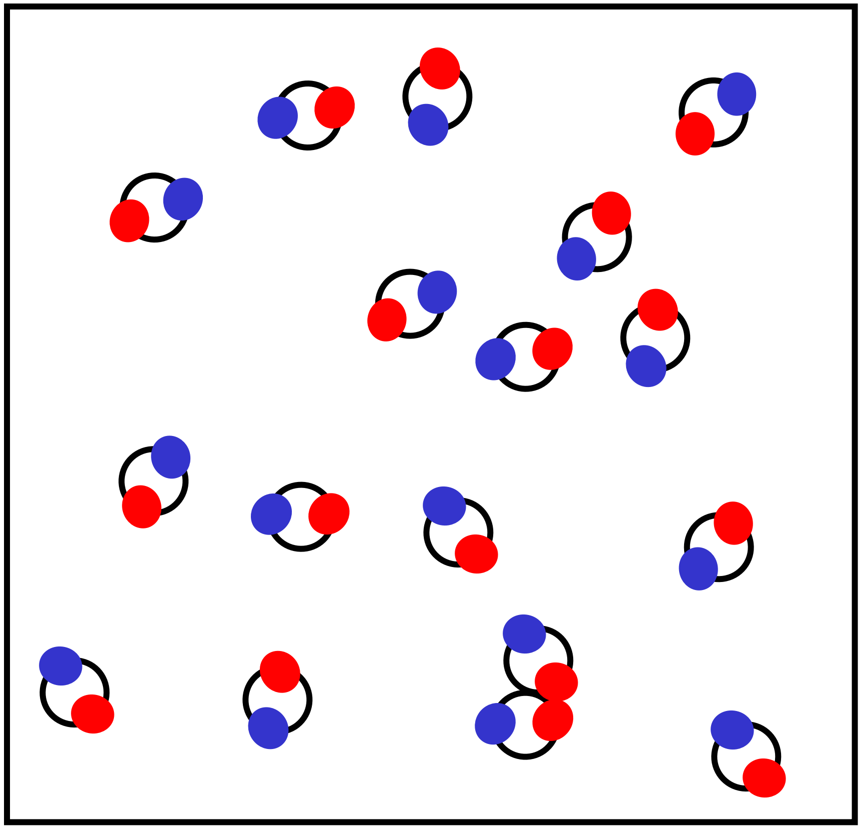
BEC of Molecules:
Condensation of
tightly bound fermion pairs

Molecules are unstable

**Atoms attract each other**
**a<0**

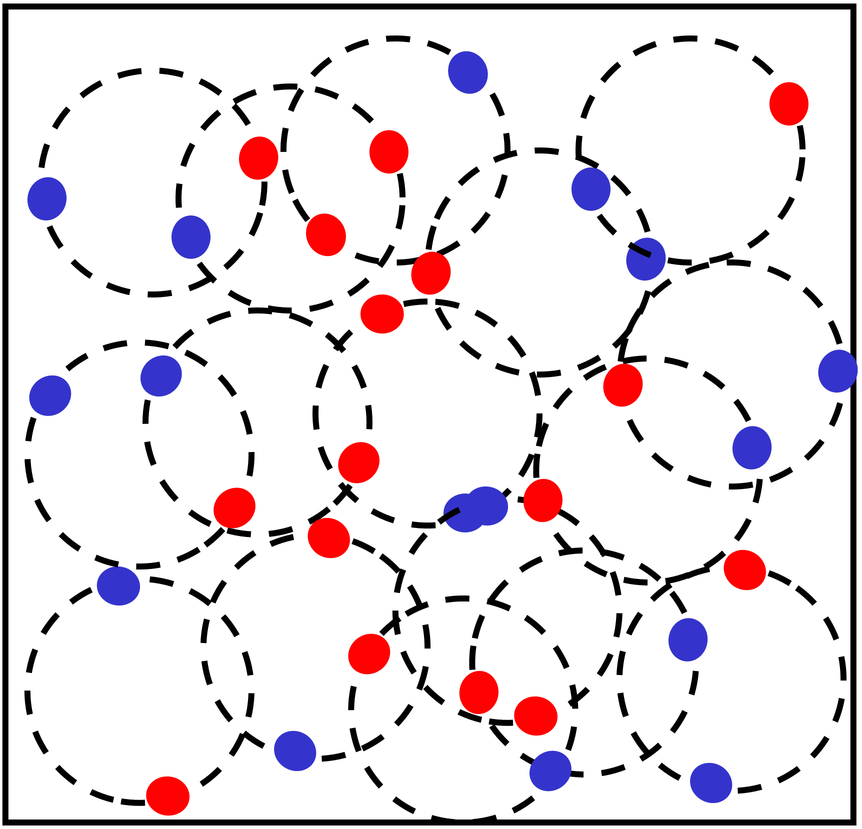
BCS-limit:
Condensation of
long-range Cooper pairs

Atom pairs

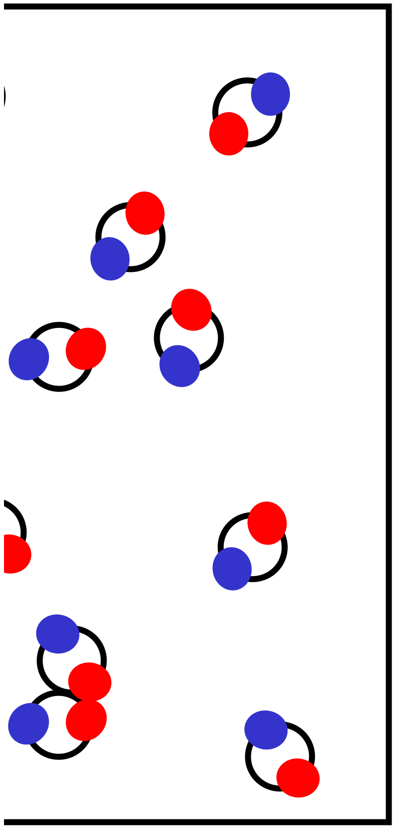


Bose Einstein condensate of molecules

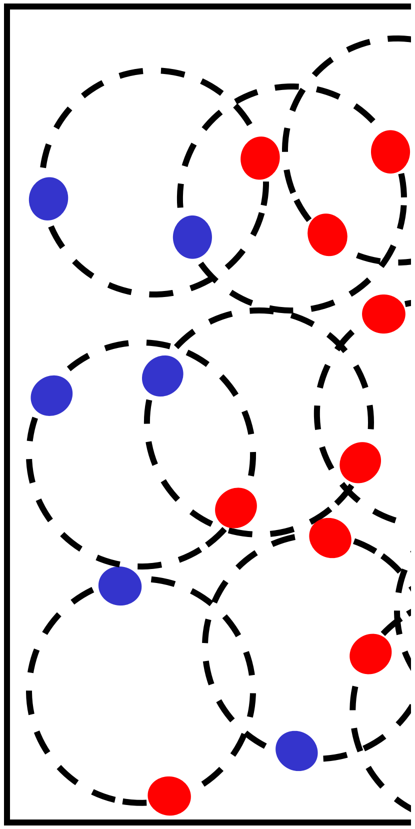
Electron pairs



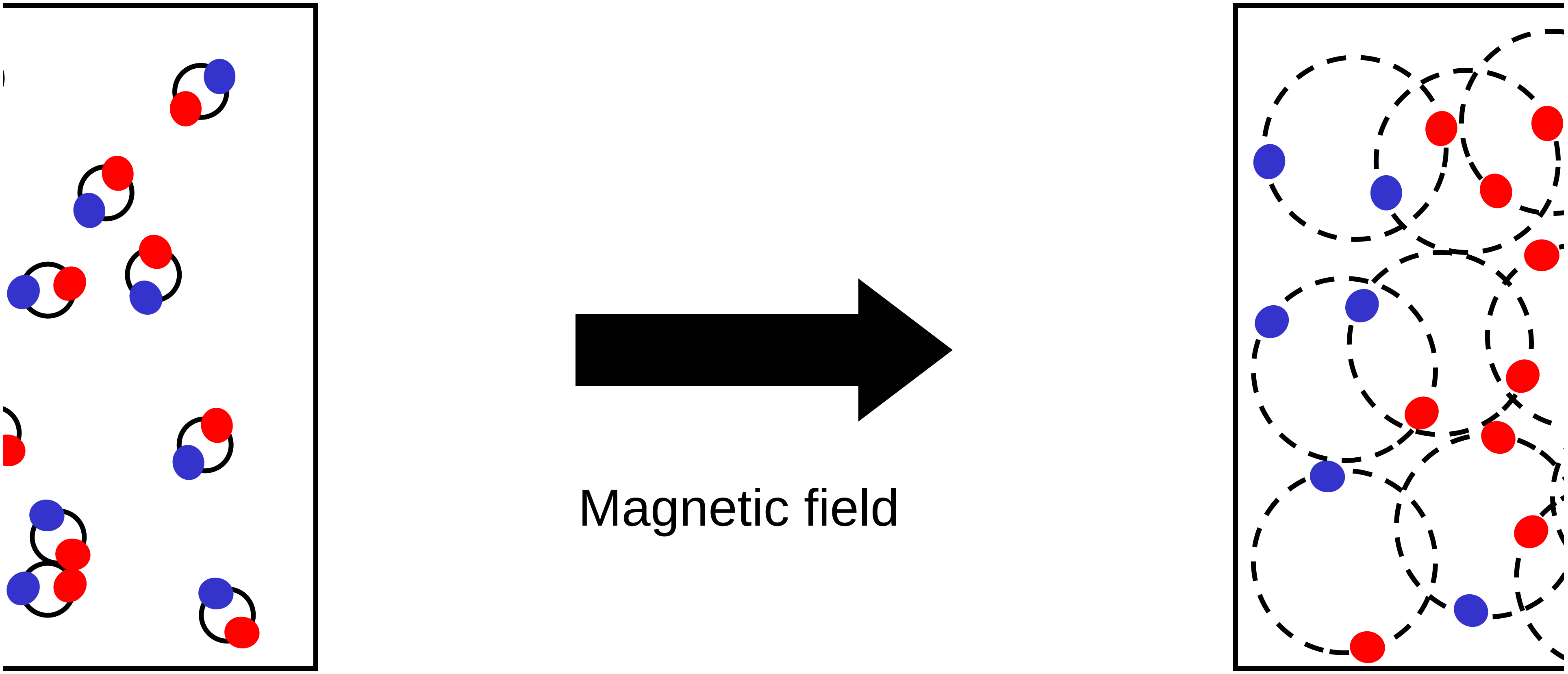
BCS Superconductor



EC



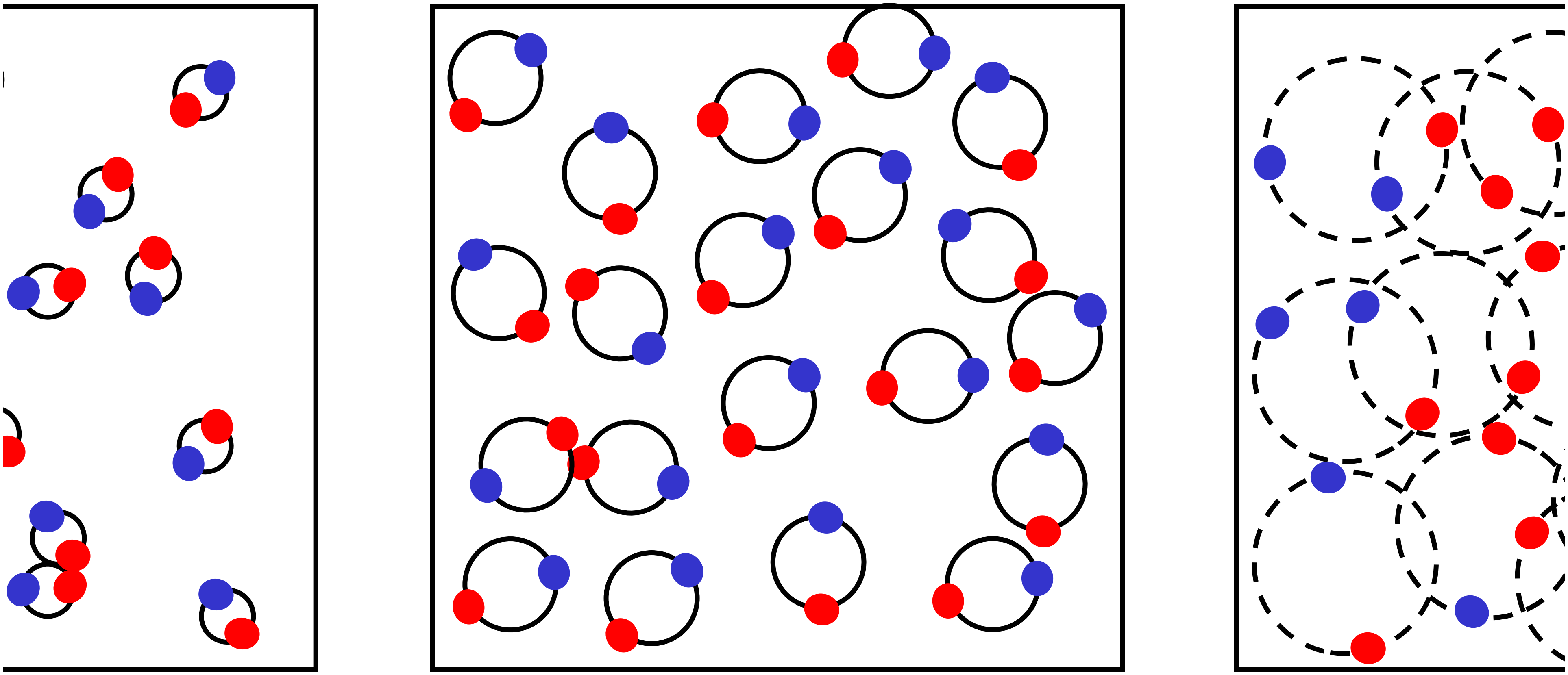
BCS sup



BEC

Magnetic field

BCS sup



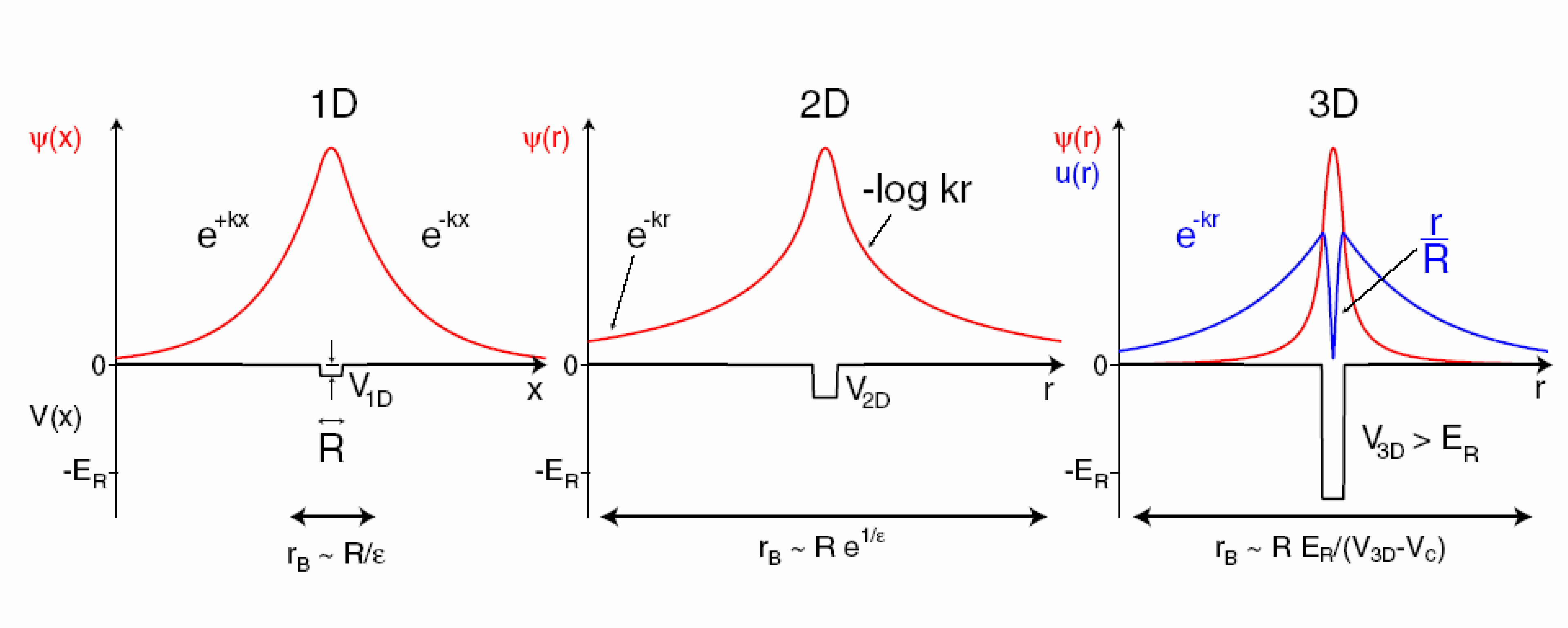
BEC

Crossover superfluid

BCS sup

# How do atoms pair?

# Two-body bound states in 1D, 2D, and 3D



1D, 2D: bound state for arbitrarily small attractive well
3D: Well depth has be larger than threshold

# Connection to the density of states

$$\frac{\hbar^2}{m}(\nabla^2 - k^2)\psi = V\psi$$

In momentum space

$$\psi_{\mathbf{k}}(\mathbf{q}) = -\frac{m}{\hbar^2}\frac{1}{q^2 + k^2}\int \frac{d^n q'}{(2\pi)^n} V(\mathbf{q} - \mathbf{q}')\psi_{\mathbf{k}}(\mathbf{q}')$$

# Connection to the density of states

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Short range potential: $V(q)=V_0$ for $q<1/R$

$$\psi_{\mathbf{k}}(\mathbf{q}) = -\frac{mV_0}{\hbar^2}\frac{1}{q^2+k^2}\int_{q'<\frac{1}{R}} \frac{d^n q'}{(2\pi)^n}\psi_{\mathbf{k}}(\mathbf{q}')$$

Integrate over q, divide by common factor $\int_{q<\frac{1}{R}} \frac{d^n q}{(2\pi)^n}\psi_{\mathbf{k}}(\mathbf{q})$

$$-\frac{1}{V_0} = \frac{m}{\hbar^2}\int_{q<\frac{1}{R}} \frac{d^n q}{(2\pi)^n}\frac{1}{q^2+k^2} = \frac{1}{\Omega}\int_{\epsilon<E_R} d\epsilon\frac{\rho_n(\epsilon)}{2\epsilon+|E|}$$

Bound state for arbitrarily small $V_0$ only if integral diverges for $E \rightarrow 0$

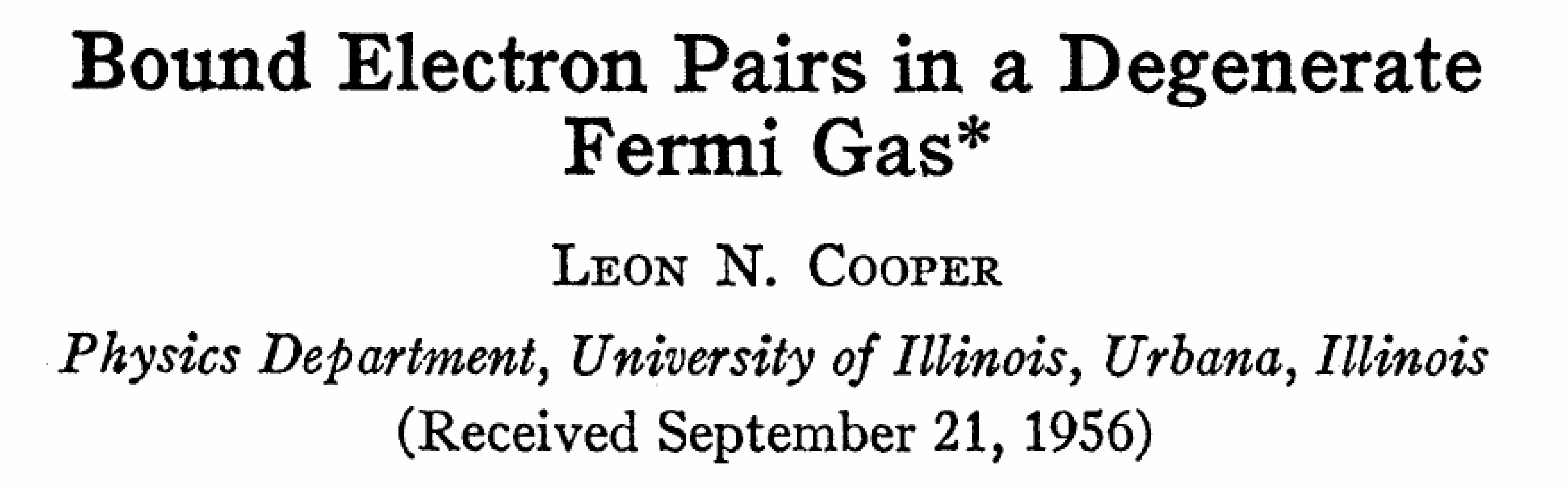
$$-\frac{1}{V_0} = \frac{m}{\hbar^2}\int_{q<\frac{1}{R}} \frac{d^n q}{(2\pi)^n}\frac{1}{q^2+k^2} = \frac{1}{\Omega}\int_{\epsilon<E_R} d\epsilon \frac{\rho_n(\epsilon)}{2\epsilon+|E|}$$

Bound state for arbitrarily small $V_0$ only if integral diverges for $E \to 0$

In 2D (constant density of states): logarithmic divergence

$$E_{2\mathrm{D}} = -2E_R\, e^{-\frac{2\Omega}{\rho_{2\mathrm{D}}|V_0|}}$$

# The Cooper problem:



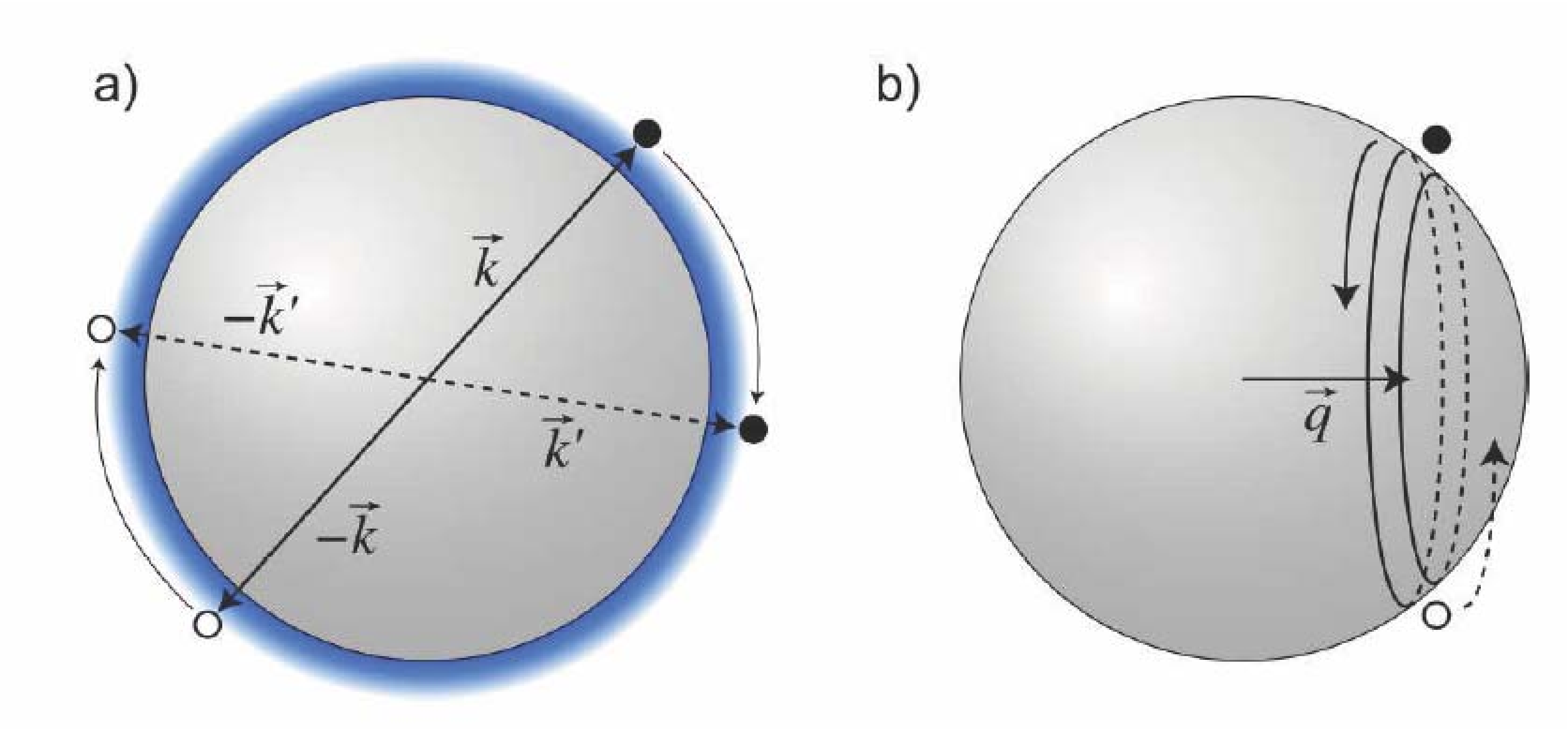
## Bound Electron Pairs in a Degenerate Fermi Gas*

LEON N. COOPER

*Physics Department, University of Illinois, Urbana, Illinois*

(Received September 21, 1956)

# Two fermions with weak interactions on top of a filled Fermi sea



Total momentum zero

Total momentum non-zero 2q

search for a small binding energy $E_B = E - 2E_F < 0$

$$-\frac{1}{V_0} = \frac{1}{\Omega} \int_{E_F < \epsilon < E_F + E_R} d\epsilon \frac{\rho_{3\mathrm{D}}(\epsilon)}{2(\epsilon - E_F) + |E_B|}$$

Pauli blocking

$$E_B = -2\, E_R \; e^{-2\Omega/\rho_{3\mathrm{D}}(E_F)|V_0|}$$

search for a small binding energy $E_B = E - 2E_F < 0$

$$-\frac{1}{V_0} = \frac{1}{\Omega} \int_{E_F < \epsilon < E_F + E_R} d\epsilon \frac{\rho_{3\mathrm{D}}(\epsilon)}{2(\epsilon - E_F) + |E_B|}$$

Pauli blocking

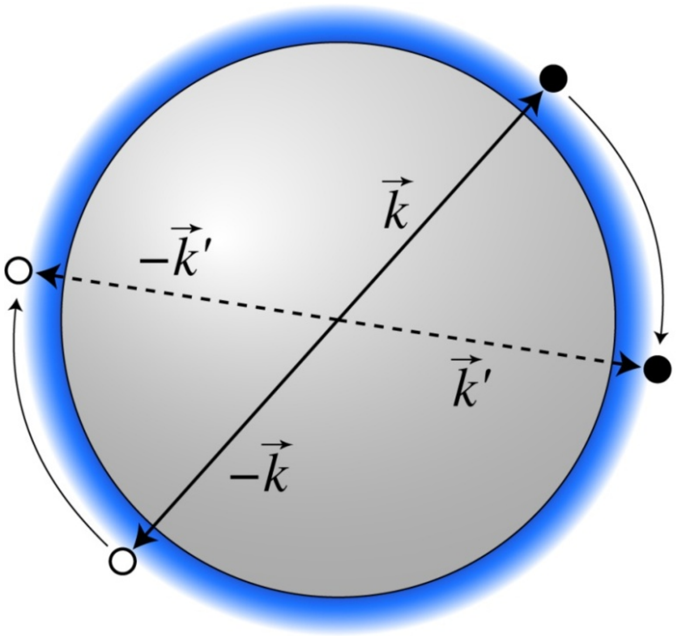
$$E_B = -2\, E_R\; e^{-2\Omega/\rho_{3\mathrm{D}}(E_F)|V_0|}$$

After replacing the bare interaction $V_0$ by the scattering length a

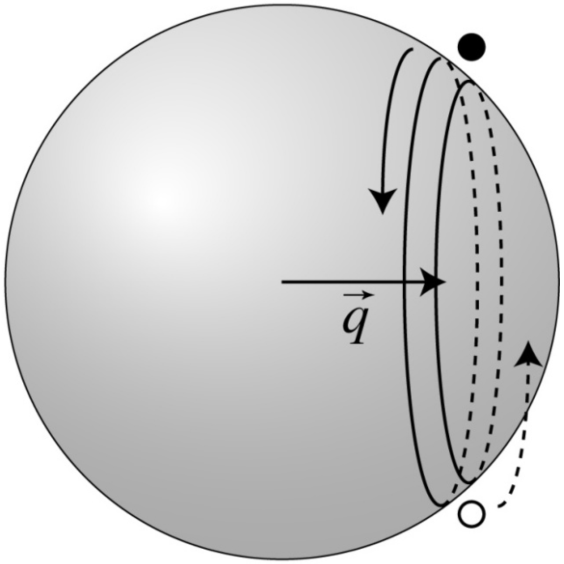
$$E_B = -\frac{8}{e^2} E_F\, e^{-\pi/k_F|a|}$$

# Cooper Pairing

Consider two particles ↑, ↓, on top of a filled, “inert” Fermi sea



Total momentum zero



Total momentum non-zero

- Reduced density of states
- Much smaller binding energy

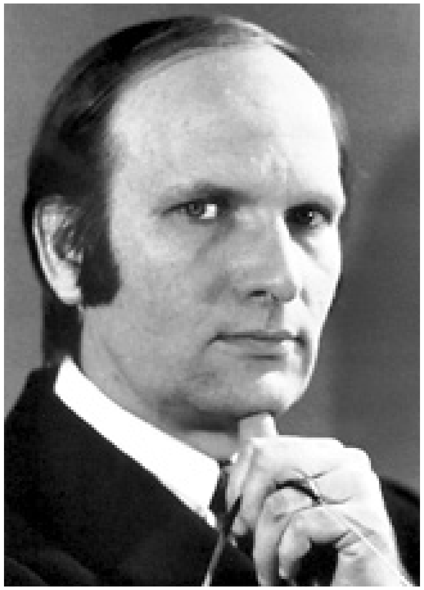
**The important pairs are those with zero momentum**

# BCS Wavefunction

**How can we find a state in which all fermions are paired in a self-consistent way?**

John Bardeen

Leon N. Cooper



John R. Schrieffer

# BCS Wavefunction

- Many-body wavefunction for a condensate of Fermion Pairs:

$$\Psi\left(\mathbf{r}_1,\ldots,\mathbf{r}_N\right) = \varphi(|\mathbf{r}_1-\mathbf{r}_2|)\chi_{12}\ldots\varphi(|\mathbf{r}_{N-1}-\mathbf{r}_N|)\chi_{N-1,N}$$

Spatial pair wavefunction

Spin wavefunction

$$\chi_{ij} = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_i\,|\downarrow\rangle_j - |\downarrow\rangle_i\,|\uparrow\rangle_j\right)$$

- Second quantization:

$$|\Psi\rangle_N = \int\prod_i d^3r_i\,\varphi(\mathbf{r}_1-\mathbf{r}_2)\Psi^\dagger_\uparrow(\mathbf{r}_1)\Psi^\dagger_\downarrow(\mathbf{r}_2)\ldots\varphi(\mathbf{r}_{N-1}-\mathbf{r}_N)\Psi^\dagger_\uparrow(\mathbf{r}_{N-1})\Psi^\dagger_\downarrow(\mathbf{r}_N)\,|0\rangle$$

- Fourier transform: Pair wavefunction: $\varphi(\mathbf{r}) = \sum_k \varphi_k \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{\Omega}}$

  Operators: $\Psi^\dagger_\sigma(\mathbf{r}) = \sum_k c^\dagger_{k\sigma}\frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{\Omega}}$

- Pair creation operator: $b^\dagger = \sum \varphi_k c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}$

- Many-body wavefunction: $|\Psi\rangle_N = b^{\dagger\,N/2}\,|0\rangle$

  *a fermion pair condensate*

# $|\Psi\rangle_N$ is not a Bose condensate

$$|\Psi\rangle_N = {b^\dagger}^{N/2} |0\rangle$$

- Commutation relations for pair creation/annihilation operators

$$[b^\dagger, b^\dagger]_- = \sum_{kk'} \varphi_k \varphi_{k'} \left[ c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}, c^\dagger_{k'\uparrow} c^\dagger_{-k'\downarrow} \right]_- = 0$$ ✓

$$[b, b]_- = \cdots = 0$$ ✓

$$[b, b^\dagger]_- = \cdots = \sum_k |\varphi_k|^2 (1 - n_{k\uparrow} - n_{k\downarrow}) \neq 1$$ ✗

Occupation of momentum $k$

- pairs do not obey Bose commutation relations, *unless* $n_k \ll 1$

$$[b, b^\dagger]_- \approx \sum_k |\varphi_k|^2 = 1$$

BEC limit of
tightly bound molecules

# BCS Wavefunction

- Introduce coherent state / switch to grand-canonical description:

$$\mathcal{N}\,|\Psi\rangle = \sum_{J_{\text{even}}} \frac{N_p^{J/4}}{(J/2)!}\,|\Psi\rangle_J = \sum_M \frac{1}{M!} N_p^{M/2}\, b^{\dagger\, M}\,|0\rangle$$

$$= e^{\sqrt{N_p}\, b^\dagger}\,|0\rangle$$

$$= \prod_k e^{\sqrt{N_p}\,\varphi_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}}\,|0\rangle$$

$c_k^\dagger$ and $c_{k'}^\dagger$ commute

$$= \prod_k \left(1 + \sqrt{N_p}\,\varphi_k\, c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\right)|0\rangle$$

because $c_k^{\dagger 2} = 0$

- Normalization: $\mathcal{N} = \prod_k \frac{1}{u_k} = \prod_k \sqrt{1 + N_p|\varphi_k|^2}$

- BCS wavefunction: $$\boxed{|\Psi_{\text{BCS}}\rangle = \prod_k \left(u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}\right)|0\rangle}$$

with $v_k = \sqrt{N_p}\,\varphi_k u_k$ and $|u_k|^2 + |v_k|^2 = 1$

# Many-Body Hamiltonian

- Second quantized Hamiltonian for interacting fermions:

$$\hat{H} = \sum_{\sigma} \int \mathrm{d}^3 r \hat{\Psi}^{\dagger}_{\sigma}(\mathbf{r}) \left( -\frac{\hbar^2 \nabla^2}{2m} \right) \hat{\Psi}_{\sigma}(\mathbf{r}) + \int \mathrm{d}^3 r \int \mathrm{d}^3 r' \hat{\Psi}^{\dagger}_{\uparrow}(\mathbf{r}) \hat{\Psi}^{\dagger}_{\downarrow}(\mathbf{r}') V(\mathbf{r} - \mathbf{r}') \hat{\Psi}_{\downarrow}(\mathbf{r}') \hat{\Psi}_{\uparrow}(\mathbf{r})$$

- Contact interaction: $V(\mathbf{r}) = V_0 \delta(\mathbf{r})$
- Fourier transform via $\Psi^{\dagger}_{\sigma}(\mathbf{r}) = \sum_k c^{\dagger}_{k\sigma} \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{\sqrt{\Omega}}$

$$\hat{H} = \sum_{k,\sigma} \epsilon_k c^{\dagger}_{k\sigma} c_{k\sigma} + \frac{V_0}{\Omega} \sum_{k,k',q} c^{\dagger}_{k+\frac{q}{2}\uparrow} c^{\dagger}_{-k+\frac{q}{2}\downarrow} c_{k'+\frac{q}{2}\downarrow} c_{-k'+\frac{q}{2}\uparrow}$$

- BCS Approximation:

  Only include scattering between zero-momentum pairs

$$\hat{H} = \sum_{k,\sigma} \epsilon_k c^{\dagger}_{k\sigma} c_{k\sigma} + \frac{V_0}{\Omega} \sum_{k,k'} c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} c_{k'\downarrow} c_{-k'\uparrow}$$

- Solve via 1) Variational Ansatz, 2) via Bogoliubov transformation

# Variational Ansatz:

- Insert BCS wavefunction into Many-Body Hamiltonian.
- Minimize Free Energy:

$$\mathcal{F} = \left\langle \hat{H} - \mu \hat{N} \right\rangle \quad = \quad \sum_k 2\xi_k v_k^2 + \frac{V_0}{\Omega} \sum_{k,k'} u_k v_k u_{k'} v_{k'}$$

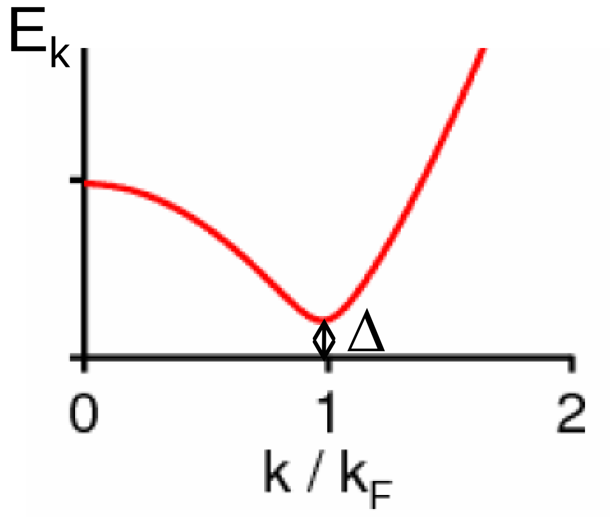
$$\text{with} \quad \xi_k = \epsilon_k - \mu$$

- Result:

$$v_k^2 \quad = \quad \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)$$

$$u_k^2 \quad = \quad \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right)$$

$$\text{with} \quad E_k \quad = \quad \sqrt{\xi_k^2 + \Delta^2}$$



- Gap equation:

$$\Delta = -\frac{V_0}{\Omega} \sum_k u_k v_k = -\frac{V_0}{\Omega} \sum_k \frac{\Delta}{2E_k}$$

# Solution via Bogoliubov Transform

- BCS Hamiltonian is quartic:

$$\hat{H} = \sum_{k,\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \frac{V_0}{\Omega} \sum_{k,k'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{k'\downarrow} c_{-k'\uparrow}$$

- Introduce pairing field (mean field or decoupling approximation):

$$C_k = \langle c_{k\uparrow} c_{-k\downarrow} \rangle$$

$$c_{k\uparrow} c_{-k\downarrow} = C_k + \underbrace{(c_{k\uparrow} c_{-k\downarrow} - C_k)}_{\text{small fluctuations (assumption)}}$$

- Neglect products (correlations) of those small fluctuations
- Define

$$\Delta = \frac{V_0}{\Omega} \sum_k C_k$$

This plays the role of the condensate wavefunction

# Solution via Bogoliubov Transform

- Rewrite Hamiltonian, drop terms quadratic in *C*’s:

$$\hat{H} = \sum_k \epsilon_k (c^\dagger_{k\uparrow} c_{k\uparrow} + c^\dagger_{k\downarrow} c_{k\downarrow}) - \Delta \sum_k \left( c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + c_{k\downarrow} c_{-k\uparrow} + \sum_{k'} C_{k'} \right)$$

Hamiltonian is now bilinear

- Solve via Bogoliubov transformation to quasiparticle operators:

$$\begin{aligned} \gamma_{k\uparrow} &= u_k c_{k\uparrow} - v_k c^\dagger_{-k\downarrow} \\ \gamma^\dagger_{-k\downarrow} &= u_k c^\dagger_{-k\downarrow} + v_k c_{k\uparrow} \end{aligned}$$

- With the choice $v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)$ and $u_k^2 = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right)$ we get

$$\text{with} \quad E_k = \sqrt{\xi_k^2 + \Delta^2}$$

$$\hat{H} - \mu\hat{N} = \underbrace{-\frac{\Delta^2}{V_0/\Omega} + \sum_k (\xi_k - E_k)}_{\text{Ground state energy}} + \underbrace{\sum_k E_k (\gamma^\dagger_{k\uparrow}\gamma_{k\uparrow} + \gamma^\dagger_{k\downarrow}\gamma_{k\downarrow})}_{\text{Non-interacting gas of fermionic quasi-particles}}$$

$$\gamma_{k\uparrow} \left|\Psi\right\rangle = 0$$

# Solution of the gap equation

$$\Delta \equiv \frac{V_0}{\Omega}\sum_k \langle c_{k\uparrow}c_{-k\downarrow}\rangle = -\frac{V_0}{\Omega}\sum_k u_k v_k = -\frac{V_0}{\Omega}\sum_k \frac{\Delta}{2E_k}$$

$$\begin{aligned} \Delta &= -\frac{V_0}{\Omega}\sum_k \frac{\Delta}{2E_k} \\ 1 &= -\frac{V_0}{\Omega}\sum_k \frac{1}{2E_k} \\ -\frac{\Omega}{V_0} &= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2\sqrt{(\epsilon_k-\mu)^2+\Delta^2}} \end{aligned}$$

$$\boxed{-\frac{\Omega}{V_0} = \int d\epsilon \, \frac{\rho_3(\epsilon)}{2\sqrt{(\epsilon-\mu)^2+\Delta^2}}}$$

Looks similar to equation for bound state and Cooper problem

# Solution of the gap equation

- Gap equation:

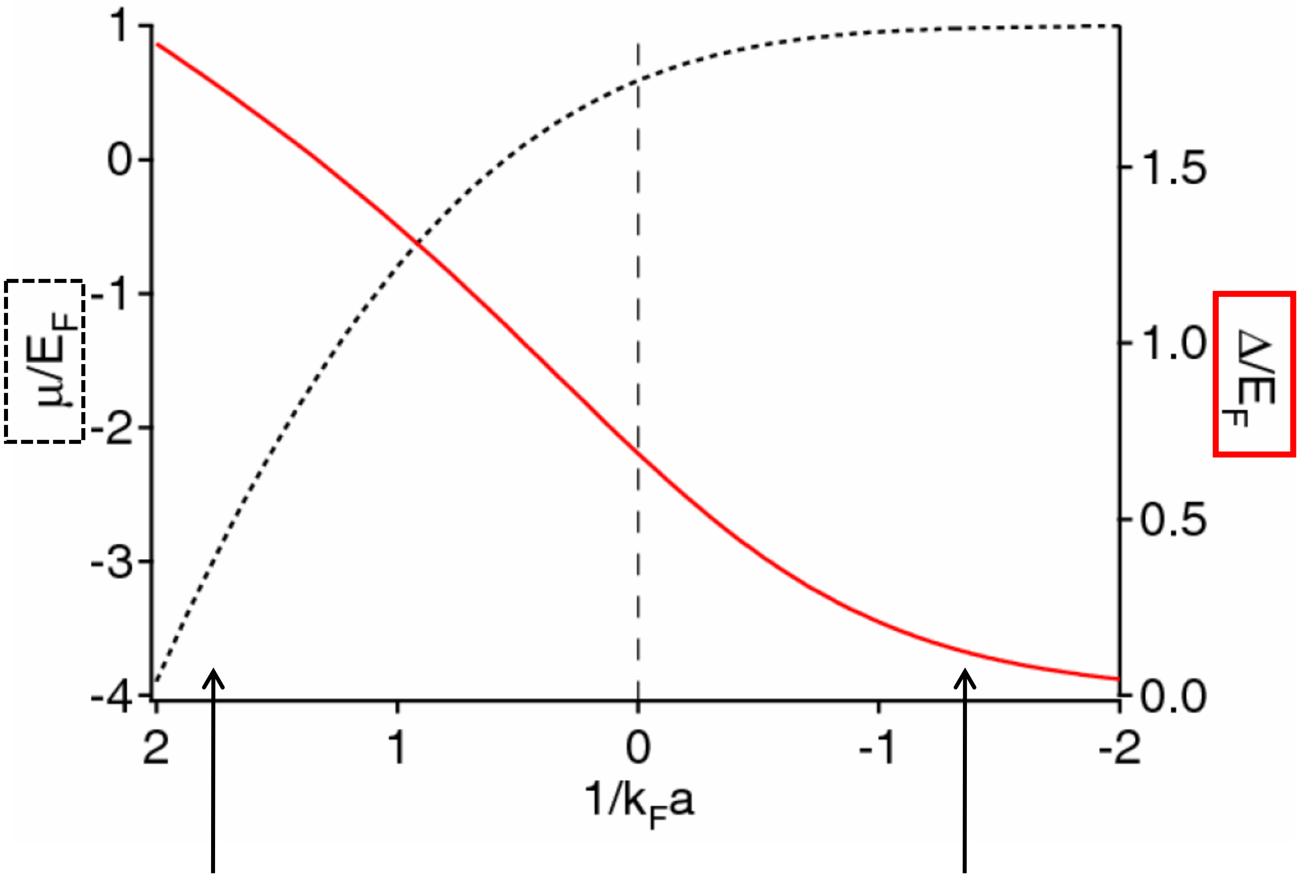
$$-\frac{\Omega}{V_0} \quad = \quad \int d\epsilon \, \frac{\rho_3(\epsilon)}{2\sqrt{(\epsilon - \boxed{\mu})^2 + \boxed{\Delta}^2}}$$

- Number equation:

$$n \quad = \quad < \hat{n} >= \sum_{k,\sigma} \left\langle c^{\dagger}_{k,\sigma} c_{k,\sigma} \right\rangle$$

- Simultaneously solve for μ and Δ

# Solution of the gap equation



BEC-side: Molecules

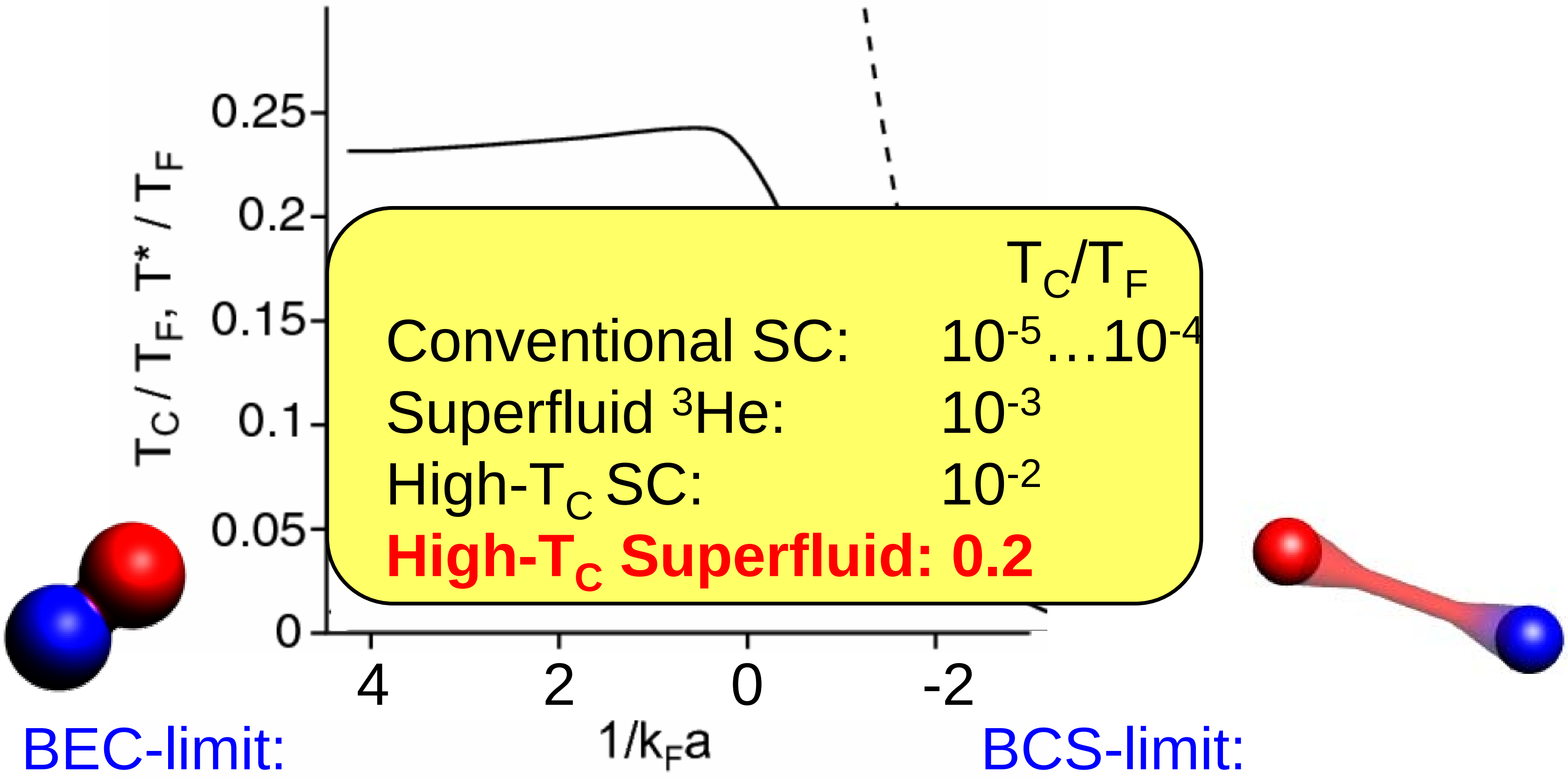
$$\mu \approx -\frac{\hbar^2}{2ma^2} = \frac{1}{2}E_B$$

BCS-side: Gap exponentially small

$$\begin{aligned}\mu &\approx E_F \\ \Delta &\approx \frac{8}{e^2}e^{-\pi/2k_F|a|}\end{aligned}$$

# Critical temperature

- Can be derived from Bogoliubov Hamiltonian with fluctuations



| | $T_C/T_F$ |
|---|---|
| Conventional SC: | $10^{-5}$…$10^{-4}$ |
| Superfluid $^3$He: | $10^{-3}$ |
| High-$T_C$ SC: | $10^{-2}$ |
| **High-$T_C$ Superfluid: 0.2** | |

BEC-limit:

$$n_{\rm Mol}\lambda^3_{\rm dB,Mol} \approx 2.612$$

$$k_B T_C = 0.22 E_F$$

BCS-limit:

$$k_B T_C = \frac{e^\gamma}{\pi}\frac{8}{e^2}e^{-\pi/2k_F|a|}$$

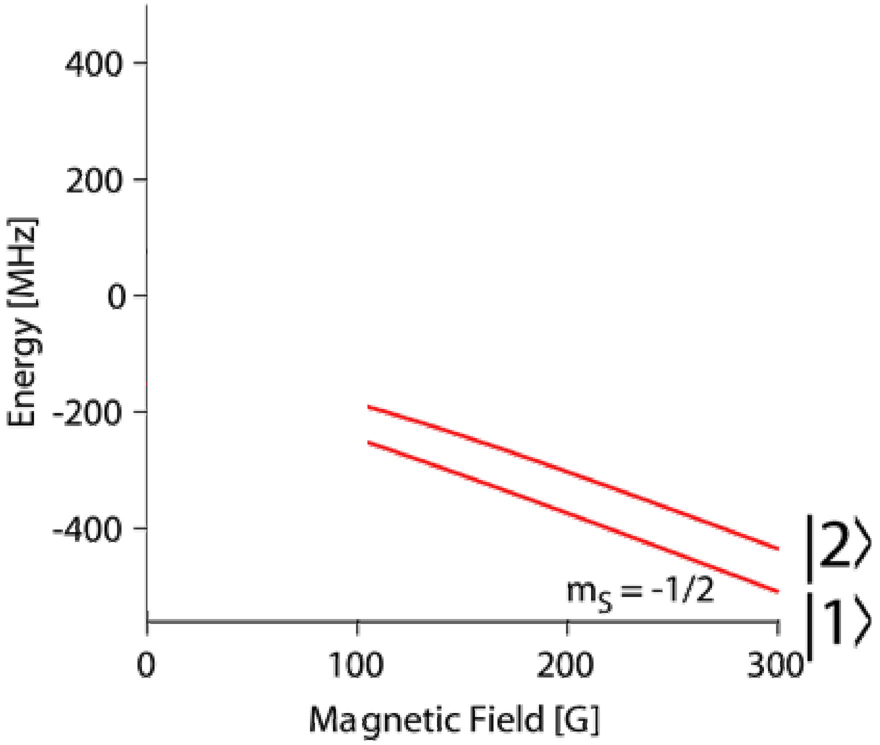
$$= \frac{e^\gamma}{\pi}\Delta_0 = 0.57\Delta_0$$

# Experimental realization of the BEC-BCS Crossover

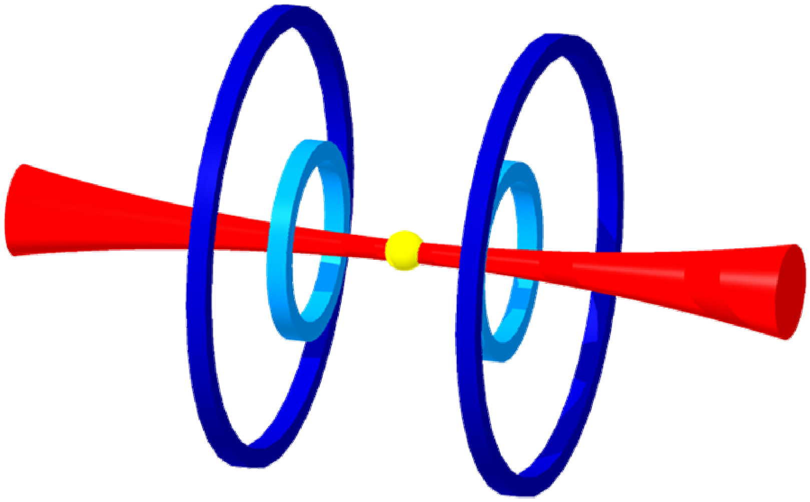
# Preparation of an interacting Fermi system in Lithium-6

Electronic spin: S = ½, Nuclear Spin: I = 1

→ (2I+1)(2S+1) = 6 hyperfine states



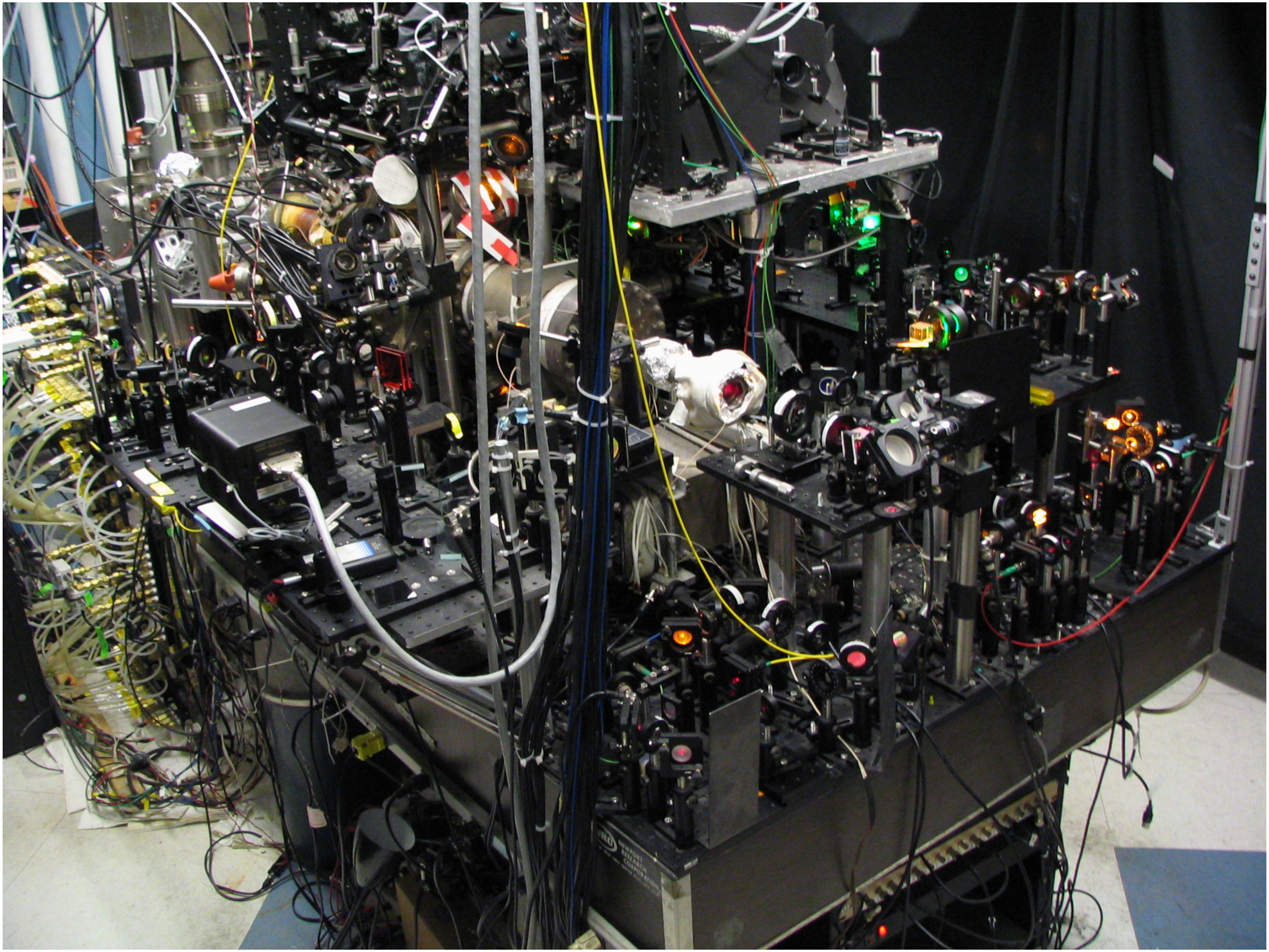
Optical trapping @ 1064 nm



$\nu_{axial}$ = 10-20 Hz

$\nu_{radial}$= 50–200 Hz

$E_{trap}$ = 0.5 - 5 μK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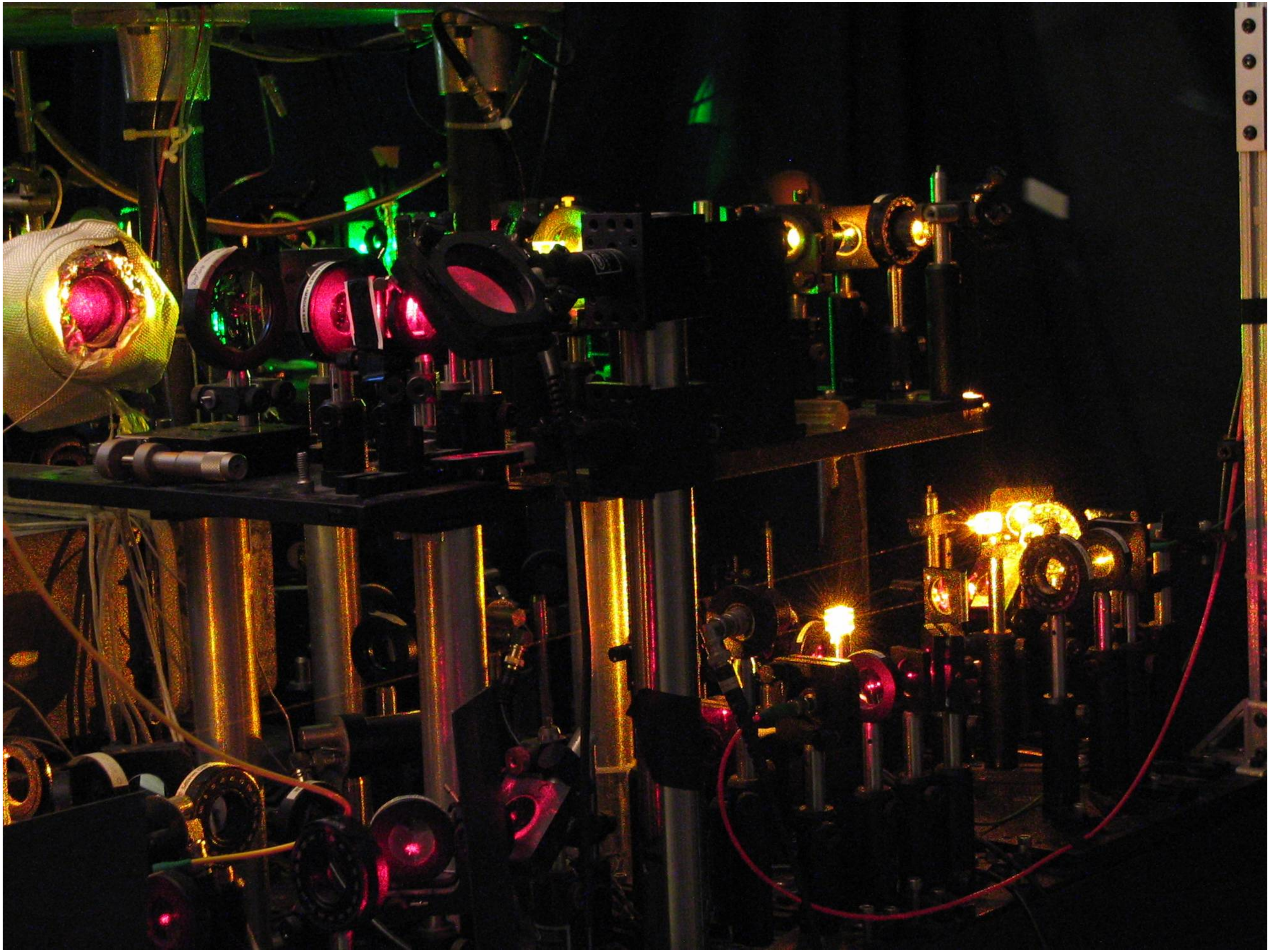
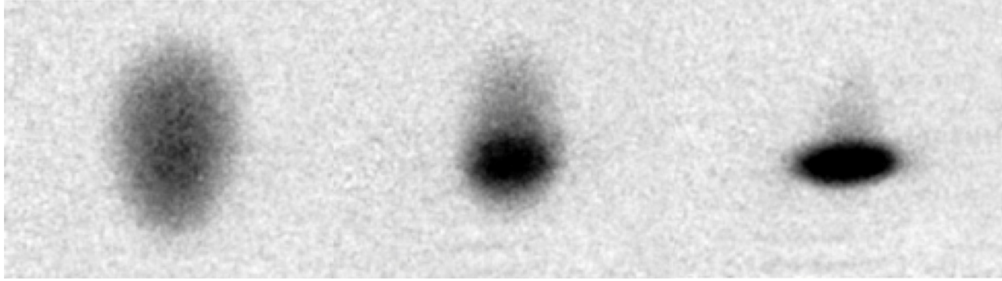


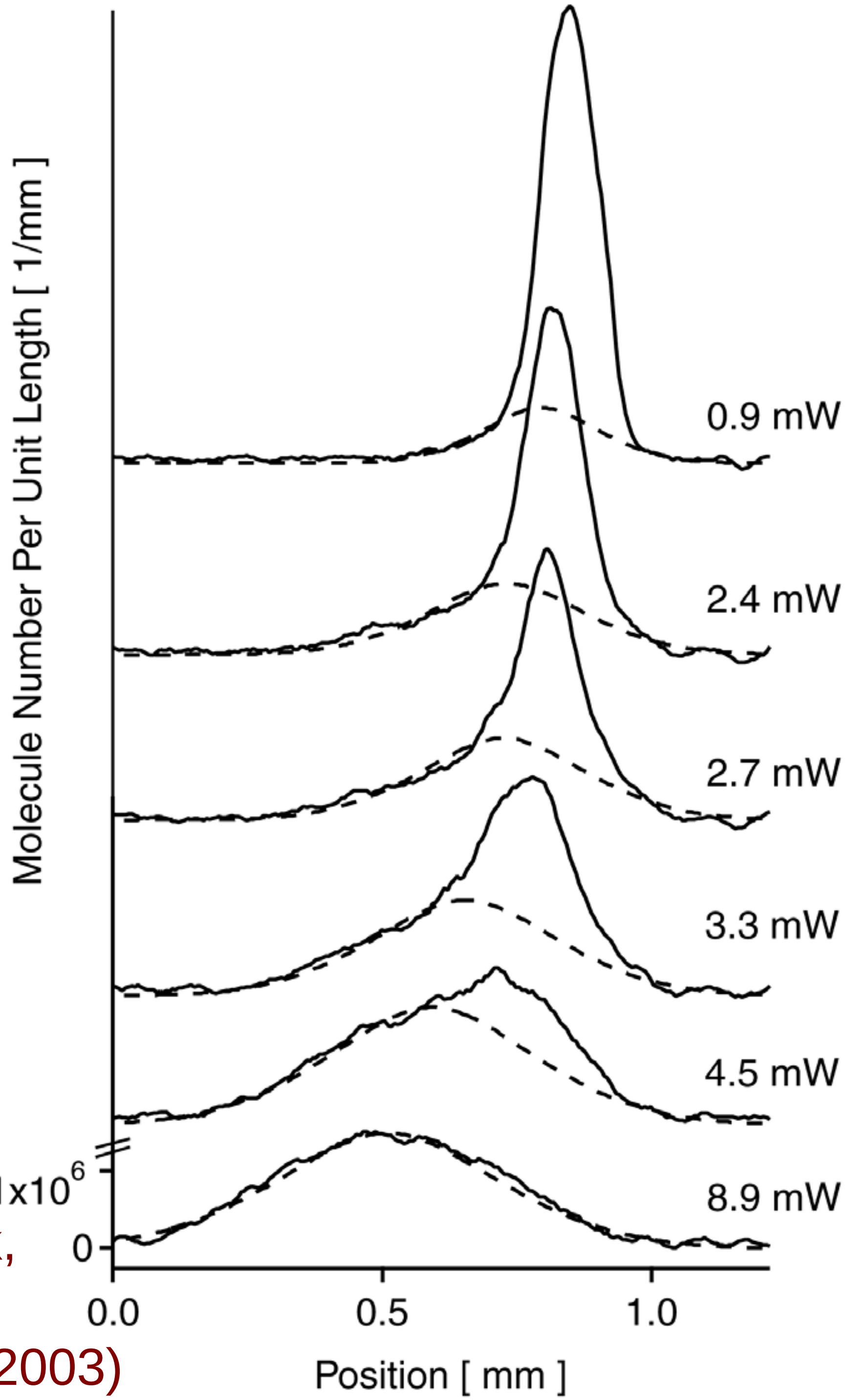
# BEC of Fermion Pairs (Molecules)



$T > T_C$ $T < T_C$ $T \ll T_C$

These days: Up to 10 million condensed molecules

| | |
|---|---|
| Boulder | Nov ‘03 |
| Innsbruck | Nov ‘03, Jan ’04 |
| MIT | Nov ’03 |
| Paris | March ’04 |
| Rice, Duke | |



M.W. Zwierlein, C. A. Stan, C. H. Schunck, S.M.F. Raupach, S. Gupta, Z. Hadzibabic, W. Ketterle, Phys. Rev. Lett. 91, 250401 (2003)

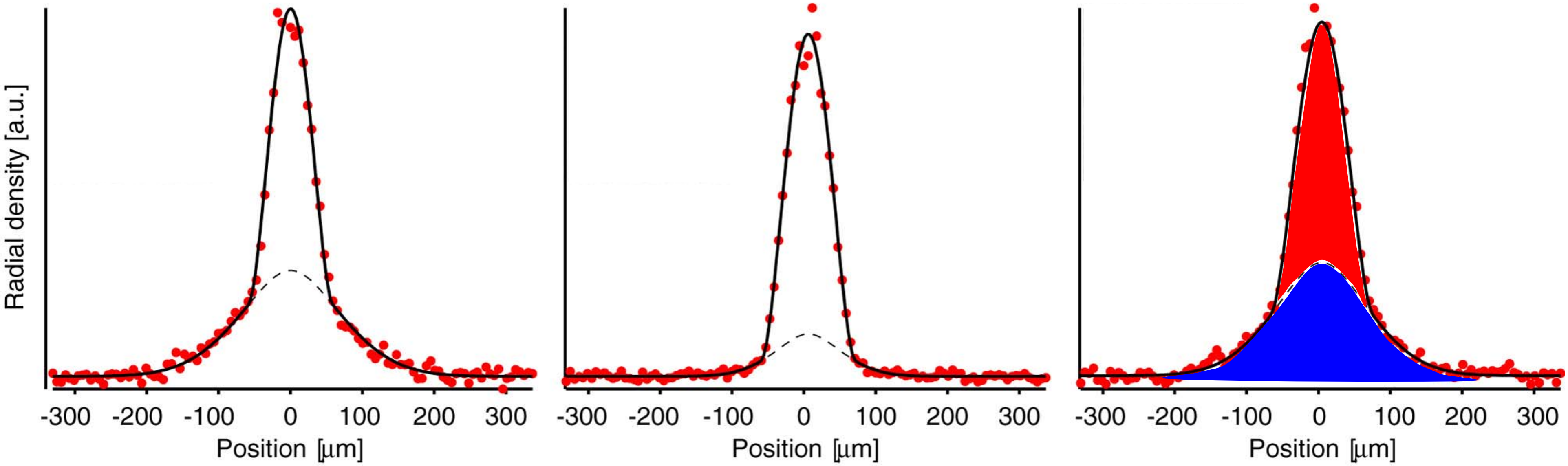
# Observation of Pair Condensates

BEC-Side

Resonance

BCS-Side

(above dissociation limit for molecules)

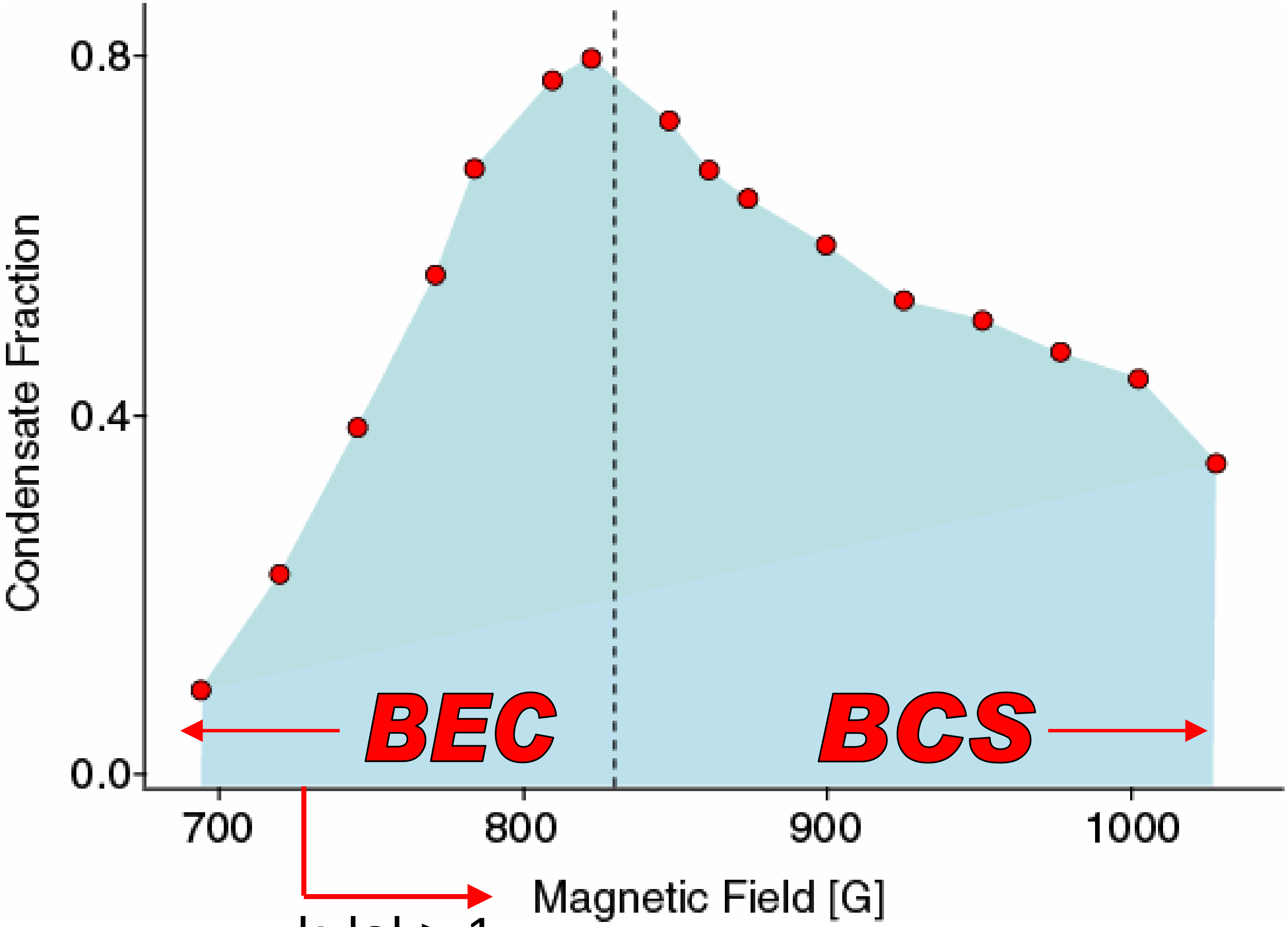


Thermal + condensed pairs

First observation: C.A. Regal et al., Phys. Rev. Lett. **92**, 040403 (2004)

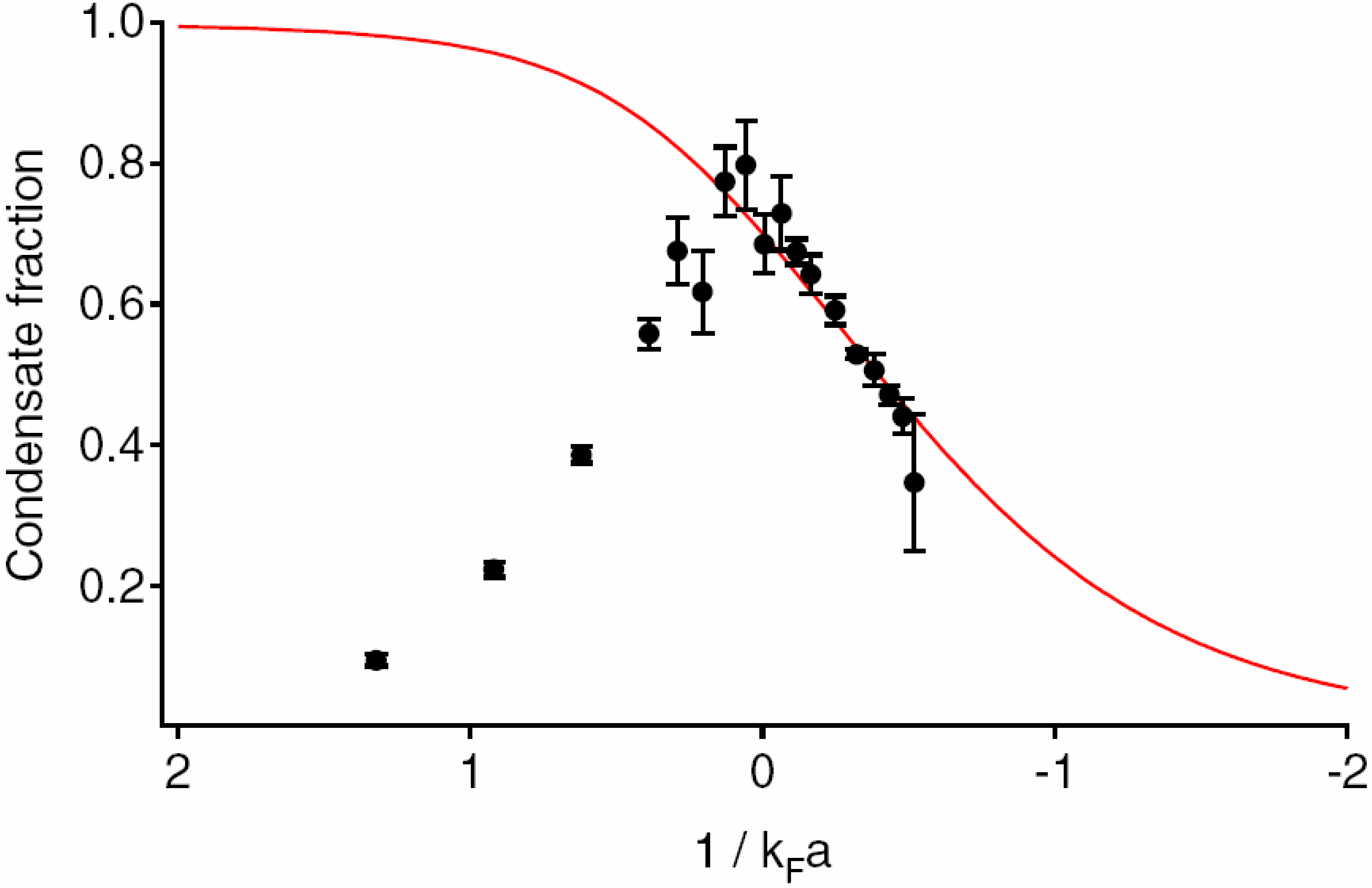
M.W. Zwierlein, C.A. Stan, C.H. Schunck, S.M.F. Raupach, A.J. Kerman, W. Ketterle, Phys. Rev. Lett. **92**, 120403 (2004).

# Condensate Fraction vs Magnetic Field



$k_F|a| > 1$

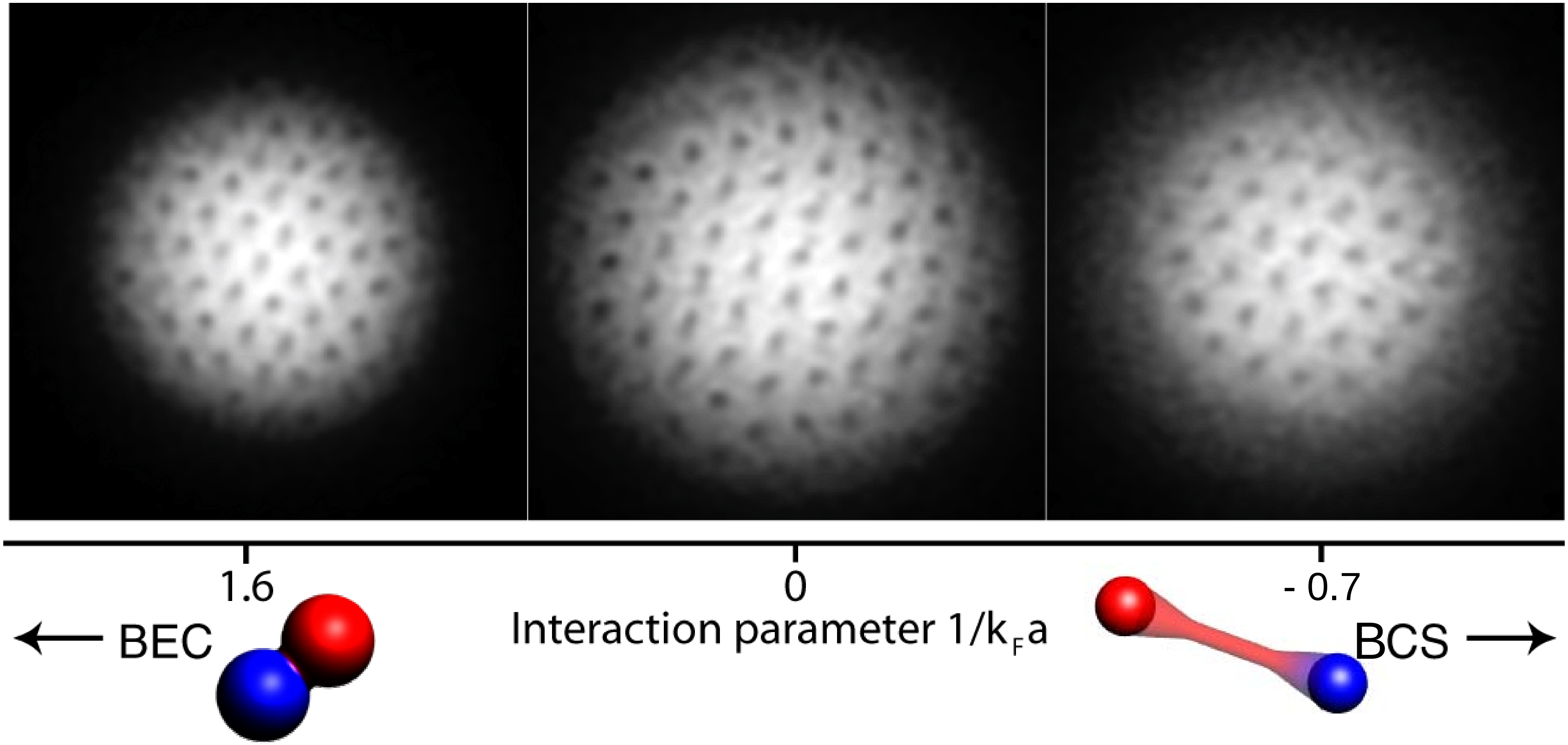
M.W. Zwierlein, C.A. Stan, C.H. Schunck, S.M.F. Raupach, A.J. Kerman,
W. Ketterle, Phys. Rev. Lett. **92**, 120403 (2004).


Condensate fraction
1.0
0.8
0.6
0.4
0.2
2
1
0
-1
-2
$1 / k_F a$

# How can we show that these gases are superfluid?

# Vortex lattices in the BEC-BCS crossover

Establishes *superfluidity* and *phase coherence* in gases of fermionic atom pairs



M.W. Zwierlein, J.R. Abo-Shaeer, A. Schirotzek, C.H. Schunck, W. Ketterle, Nature 435, 1047-1051 (2005)